Problems & Solutions

1. *Anisotropic nonlinear σ model:* Consider unit n-component spins, $\vec{s}(\mathbf{x}) = (s_1, \cdots, s_n)$ with $\sum_\alpha s_\alpha^2 = 1$, subject to a Hamiltonian

$$\beta\mathcal{H} = \int d^d\mathbf{x} \left[\frac{1}{2T} \left(\nabla \vec{s}\right)^2 + g s_1^2 \right].$$

For $g = 0$, renormalization group equations are obtained through rescaling distances by a factor $b = e^\ell$, and spins by a factor $\zeta = b^{y_s}$ with $y_s = -\frac{(n-1)}{4\pi}T$, leading to (for $\epsilon = d - 2$)

$$\frac{dT}{d\ell} = -\epsilon T + \frac{(n-2)}{2\pi}T^2 + \mathcal{O}(T^3)\,.$$

(a) Find the fixed point, and the thermal eigenvalue y_T.

• Setting $dT/d\ell$ to zero, the fixed point is obtained as

$$T^* = \frac{2\pi\epsilon}{n-2} + \mathcal{O}(\epsilon^2).$$

Linearizing the recursion relation gives

$$y_T = -\epsilon + \frac{(n-2)}{\pi}T^* = +\epsilon + \mathcal{O}(\epsilon^2).$$

(b) Write the renormalization group equation for g in the vicinity of the above fixed point, and obtain the corresponding eigenvalue y_g.

• Rescalings $x \to b\mathbf{x}'$ and $\vec{s} \to \zeta\vec{s}\,'$, lead to $g \to g' = b^d\zeta^2 g$, and hence

$$y_g = d + 2y_s = d - \frac{n-1}{2\pi}T^* = 2 + \epsilon - \frac{n-1}{n-2}\epsilon = 2 - \frac{1}{n-2}\epsilon + \mathcal{O}(\epsilon^2).$$

(c) Sketch the phase diagram as a function of T and g, indicating the phases, and paying careful attention to the shape of the phase boundary as $g \to 0$.

• The term proportional to g removes full rotational symmetry and leads to a bicritical phase diagram as discussed in recitations. The phase for $g < 0$ has order along direction 1, while $g > 0$ favors ordering along any one of the $(n-1)$ directions orthogonal to 1. The phase boundaries as $g \to 0$ behave as $g \propto (\delta T)^\phi$, with $\phi = y_g/y_t \approx 2/\epsilon + \mathcal{O}(1)$.

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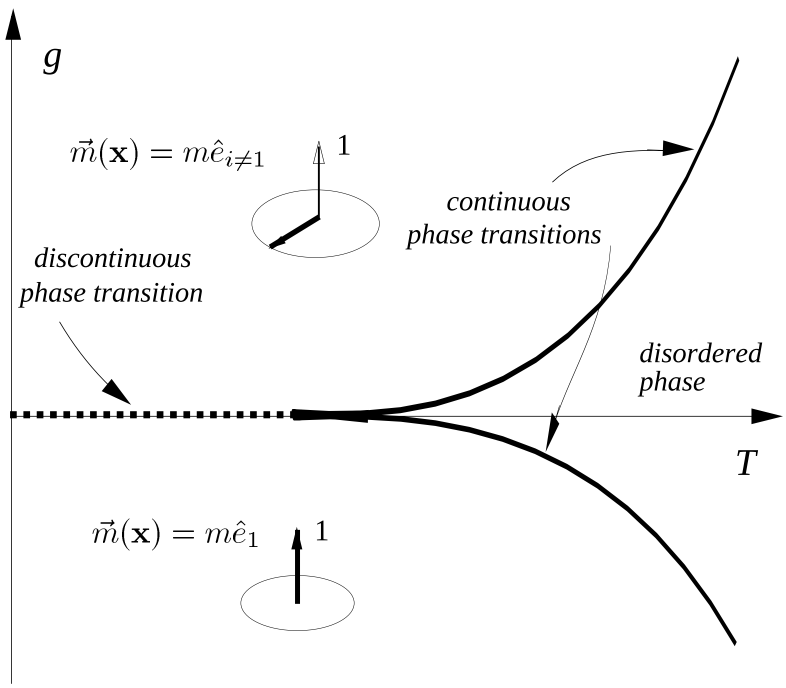

2. *Matrix models:* In some situations, the order parameter is a matrix rather than a vector. For example, in triangular (Heisenberg) antiferromagnets each triplet of spins aligns at 120°, locally defining a plane. The variations of this plane across the system are described by a 3×3 rotation matrix. We can construct a nonlinear σ model to describe a generalization of this problem as follows. Consider the Hamiltonian

$$\beta\mathcal{H} = \frac{K}{4}\int d^d\mathbf{x}\ \mathrm{tr}\left[\nabla M(\mathbf{x}) \cdot \nabla M^T(\mathbf{x})\right] \quad ,$$

where M is a *real,* $N \times N$ *orthogonal* matrix, and 'tr' denotes the trace operation. The condition of orthogonality is that $MM^T = M^T M = I$, where I is the $N \times N$ identity matrix, and M^T is the transposed matrix, $M^T_{ij} = M_{ji}$. The partition function is obtained by summing over all matrix functionals, as

$$Z = \int \mathcal{D}M(\mathbf{x})\delta\left(M(\mathbf{x})M^T(\mathbf{x}) - I\right) e^{-\beta\mathcal{H}[M(\mathbf{x})]} \quad .$$

(a) Rewrite the Hamiltonian and the orthogonality constraint in terms of the matrix elements M_{ij} $(i, j = 1, \cdots, N)$. Describe the ground state of the system.

• In terms of the matrix elements, the Hamiltonian reads

$$\beta\mathcal{H} = \frac{K}{4}\int d^d x \sum_{i,j} \nabla M_{ij} \cdot \nabla M_{ij},$$

and the orthogonality condition becomes

$$\sum_k M_{ik} M_{jk} = \delta_{ij}.$$

Since $\nabla M_{ij} \cdot \nabla M_{ij} \geq 0$, any constant (spatially uniform) orthogonal matrix realizes a ground state.

(b) Define the symmetric and anti-symmetric matrices

$$\begin{cases} \sigma = \frac{1}{2}\left(M + M^T\right) = \sigma^T \\ \pi = \frac{1}{2}\left(M - M^T\right) = -\pi^T \end{cases}.$$

Express $\beta\mathcal{H}$ and the orthogonality constraint in terms of the matrices σ and π.

• As $M = \sigma + \pi$ and $M^T = \sigma - \pi$,

$$\beta\mathcal{H} = \frac{K}{4}\int d^d x \, \mathrm{tr}\left[\nabla\left(\sigma+\pi\right)\cdot\nabla\left(\sigma-\pi\right)\right] = \frac{K}{4}\int d^d x \, \mathrm{tr}\left[\left(\nabla\sigma\right)^2 - \left(\nabla\pi\right)^2\right],$$

where we have used the (easily checked) fact that the trace of the cummutator of matrices $\nabla\sigma$ and $\nabla\pi$ is zero. Similarly, the orthogonality condition is written as

$$\sigma^2 - \pi^2 = I,$$

where I is the unit matrix.

(c) Consider *small fluctuations* about the ordered state $M(\mathbf{x}) = I$. Show that σ can be expanded in powers of π as

$$\sigma = I - \frac{1}{2}\pi\pi^T + \cdots.$$

Use the orthogonality constraint to integrate out σ, and obtain an expression for βH to fourth order in π. Note that there are two distinct types of fourth order terms. *Do not include* terms generated by the argument of the delta function. As shown for the nonlinear σ model in the text, these terms do not effect the results at lowest order.

• Taking the square root of

$$\sigma^2 = I + \pi^2 = I - \pi\pi^T,$$

results in

$$\sigma = I - \frac{1}{2}\pi\pi^T + \mathcal{O}\left(\pi^4\right),$$

(as can easily be checked by calculating the square of $I - \pi\pi^T/2$). We now integrate out σ, to obtain

$$Z = \int \mathcal{D}\pi\left(\mathbf{x}\right)\exp\left\{\frac{K}{4}\int d^d x \, \mathrm{tr}\left[\left(\nabla\pi\right)^2 - \frac{1}{4}\left(\nabla\left(\pi\pi^T\right)\right)^2\right]\right\},$$

where $\mathcal{D}\pi(\mathbf{x}) = \prod_{j>i} \mathcal{D}\pi_{ij}(\mathbf{x})$, and π is a matrix with zeros along the diagonal, and elements below the diagonal given by $\pi_{ij} = -\pi_{ji}$. Note that we have not included the terms generated by the argument of the delta function. Such term, which ensure that the measure of integration over π is symmetric, do not contribute to the renormalization of K at the lowest order. Note also that the fourth order terms are of two distinct types, due to the non-commutativity of π and $\nabla\pi$. Indeed,

$$\begin{aligned}\left[\nabla\left(\pi\pi^T\right)\right]^2 &= \left[\nabla\left(\pi^2\right)\right]^2 = \left[(\nabla\pi)\pi + \pi\nabla\pi\right]^2 \\ &= (\nabla\pi)\pi\cdot(\nabla\pi)\pi + (\nabla\pi)\pi^2\cdot\nabla\pi + \pi(\nabla\pi)^2\pi + \pi(\nabla\pi)\pi\cdot\nabla\pi,\end{aligned}$$

and, since the trace is unchanged by cyclic permutations,

$$\operatorname{tr}\left[\nabla\left(\pi\pi^T\right)\right]^2 = 2\operatorname{tr}\left[(\pi\nabla\pi)^2 + \pi^2(\nabla\pi)^2\right].$$

(d) For an N-vector order parameter there are $N-1$ Goldstone modes. Show that an orthogonal $N\times N$ order parameter leads to $N(N-1)/2$ such modes.

• The anti-symmetry of π imposes $N(N+1)/2$ conditions on the $N\times N$ matrix elements, and thus there are $N^2 - N(N+1)/2 = N(N-1)/2$ independent components (Goldstone modes) for the matrix. Alternatively, the orthogonality of M similarly imposes $N(N+1)/2$ constraints, leading to $N(N-1)/2$ degrees of freedom. [Note that in the analogous calculation for the $\mathcal{O}(n)$ model, there is one condition constraining the magnitude of the spins to unity; and the remaining $n-1$ angular components are Goldstone modes.]

(e) Consider the quadratic piece of $\beta\mathcal{H}$. Show that the two point correlation function in Fourier space is

$$\langle\pi_{ij}(\mathbf{q})\pi_{kl}(\mathbf{q}')\rangle = \frac{(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')}{Kq^2}\left[\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right].$$

• In terms of the Fourier components $\pi_{ij}(\mathbf{q})$, the quadratic part of the Hamiltonian in (c) has the form

$$\beta\mathcal{H}_0 = \frac{K}{2}\sum_{i<j}\int\frac{d^d\mathbf{q}}{(2\pi)^d}q^2|\pi_{ij}(\mathbf{q})|^2,$$

leading to the bare expectation values

$$\langle\pi_{ij}(\mathbf{q})\pi_{ij}(\mathbf{q}')\rangle_0 = \frac{(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')}{Kq^2},$$

and

$$\langle\pi_{ij}(\mathbf{q})\pi_{kl}(\mathbf{q}')\rangle_0 = 0, \text{ if the pairs } (ij) \text{ and } (kl) \text{ are different.}$$

Furthermore, since π is anti-symmetric,

$$\langle \pi_{ij}(\mathbf{q})\,\pi_{ji}(\mathbf{q}')\rangle_0 = -\langle \pi_{ij}(\mathbf{q})\,\pi_{ij}(\mathbf{q}')\rangle_0\,,$$

and in particular $\langle \pi_{ii}(\mathbf{q})\,\pi_{jj}(\mathbf{q}')\rangle_0 = 0$. These results can be summarized by

$$\langle \pi_{ij}(\mathbf{q})\,\pi_{kl}(\mathbf{q}')\rangle_0 = \frac{(2\pi)^d\,\delta^d(\mathbf{q}+\mathbf{q}')}{Kq^2}\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right).$$

We shall now construct a renormalization group by removing Fourier modes $M^>(\mathbf{q})$, with $\mathbf{q}$ in the shell $\Lambda/b < |\mathbf{q}| < \Lambda$.

(f) Calculate the coarse grained expectation value for $\langle \mathrm{tr}(\sigma)\rangle_0^>$ at low temperatures after removing these modes. Identify the scaling factor, $M'(\mathbf{x}') = M^<(\mathbf{x})/\zeta$, that restores $\mathrm{tr}(M') = \mathrm{tr}(\sigma') = N$.

• As a result of fluctuations of short wavelength modes, $\mathrm{tr}\,\sigma$ is reduced to

$$\begin{aligned}
\langle \mathrm{tr}\,\sigma\rangle_0^> &= \left\langle \mathrm{tr}\left(I + \frac{\pi^2}{2} + \cdots\right)\right\rangle_0^> \approx N + \frac{1}{2}\left\langle \mathrm{tr}\,\pi^2\right\rangle_0^> \\
&= N + \frac{1}{2}\left\langle \sum_{i\neq j}\pi_{ij}\pi_{ji}\right\rangle_0^> = N - \frac{1}{2}\left\langle \sum_{i\neq j}\pi_{ij}^2\right\rangle_0^> = N - \frac{1}{2}\left(N^2 - N\right)\left\langle \pi_{ij}^2\right\rangle_0^> \\
&= N\left(1 - \frac{N-1}{2}\int_{\Lambda/b}^{\Lambda}\frac{d^dq}{(2\pi)^d}\frac{1}{Kq^2}\right) = N\left[1 - \frac{N-1}{2K}I_d(b)\right].
\end{aligned}$$

To restore $\mathrm{tr}\,M' = \mathrm{tr}\,\sigma' = N$, we rescale all components of the matrix by

$$\zeta = 1 - \frac{N-1}{2K}I_d(b)\,.$$

NOTE: An orthogonal matrix M is invertible ($M^{-1} = M^T$), and therefore diagonalizable. In diagonal form, the transposed matrix is equal to the matrix itself, and so its square is the identity, implying that each eigenvalue is either $+1$ or -1. Thus, if M is chosen to be very close to the identity, all eigenvalues are $+1$, and $\mathrm{tr}\,M = N$ (as the trace is independent of the coordinate basis).

(g) Use perturbation theory to calculate the coarse grained coupling constant $\tilde{K}$. Evaluate only the two diagrams that directly renormalize the $(\nabla\pi_{ij})^2$ term in $\beta\mathcal{H}$, and show that

$$\tilde{K} = K + \frac{N}{2}\int_{\Lambda/b}^{\Lambda}\frac{d^d\mathbf{q}}{(2\pi)^d}\frac{1}{q^2}\quad .$$

• Distinguishing between the greater and lesser modes, we write the partition function as

$$Z = \int \mathcal{D}\pi^< \mathcal{D}\pi^> \, e^{-\beta\mathcal{H}_0^< - \beta\mathcal{H}_0^> + U[\pi^<, \pi^>]} = \int \mathcal{D}\pi^< \, e^{-\delta f_b^0 - \beta\mathcal{H}_0^<} \left\langle e^U \right\rangle_0^> ,$$

where $\mathcal{H}_0$ denotes the quadratic part, and

$$\begin{aligned} U &= -\frac{K}{8} \sum_{i,j,k,l} \int d^d x \left[(\nabla \pi_{ij}) \, \pi_{jk} \cdot (\nabla \pi_{kl}) \, \pi_{li} + \pi_{ij} \, (\nabla \pi_{jk}) \cdot (\nabla \pi_{kl}) \, \pi_{li} \right] \\ &= \frac{K}{8} \sum_{i,j,k,l} \int \frac{d^d q_1 d^d q_2 d^d q_3}{(2\pi)^{3d}} \left[(\mathbf{q}_1 \cdot \mathbf{q}_3 + \mathbf{q}_2 \cdot \mathbf{q}_3) \cdot \right. \\ &\qquad\qquad \left. \cdot \pi_{ij} (\mathbf{q}_1) \, \pi_{jk} (\mathbf{q}_2) \, \pi_{kl} (\mathbf{q}_3) \, \pi_{li} (-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) \right] . \end{aligned}$$

To first order in U, the following two averages contribute to the renormalization of K:

$$\begin{aligned} (i) \qquad & \frac{K}{8} \sum_{i,j,k,l} \int \frac{d^d q_1 d^d q_2 d^d q_3}{(2\pi)^{3d}} \left\langle \pi_{jk}^> (\mathbf{q}_2) \, \pi_{li}^> (-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) \right\rangle_0^> (\mathbf{q}_1 \cdot \mathbf{q}_3) \, \pi_{ij}^< (\mathbf{q}_1) \, \pi_{kl}^< (\mathbf{q}_3) \\ = & \frac{K}{8} \left(\int_{\Lambda/b}^{\Lambda} \frac{d^d q'}{(2\pi)^d} \frac{1}{K q'^2} \right) \left(\int_0^{\Lambda/b} \frac{d^d q}{(2\pi)^d} q^2 \sum_{i,j} \pi_{ij}^< (\mathbf{q}) \, \pi_{ji}^< (-\mathbf{q}) \right) , \end{aligned}$$

and

$$\begin{aligned} (ii) \; & \frac{K}{8} \sum_{j,k,l} \int \frac{d^d q_1 d^d q_2 d^d q_3}{(2\pi)^{3d}} \left\langle \sum_{i \neq j,l} \pi_{ij}^> (\mathbf{q}_2) \, \pi_{li}^> (-\mathbf{q}_1 - \mathbf{q}_2 - \mathbf{q}_3) \right\rangle_0^> (\mathbf{q}_2 \cdot \mathbf{q}_3) \, \pi_{jk}^< (\mathbf{q}_2) \, \pi_{kl}^< (\mathbf{q}_3) \\ = & \frac{K}{8} \left[(N-1) \int_{\Lambda/b}^{\Lambda} \frac{d^d q'}{(2\pi)^d} \frac{1}{K q'^2} \right] \left(\int_0^{\Lambda/b} \frac{d^d q}{(2\pi)^d} q^2 \sum_{j,k} \pi_{jk}^< (\mathbf{q}) \, \pi_{kj}^< (-\mathbf{q}) \right) . \end{aligned}$$

Adding up the two contributions results in an effective coupling

$$\frac{\tilde{K}}{4} = \frac{K}{4} + \frac{K}{8} N \int_{\Lambda/b}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{K q^2}, \qquad i.e. \qquad \tilde{K} = K + \frac{N}{2} I_d (b) .$$

(h) Using the result from part (f), show that after matrix rescaling, the RG equation for K' is given by:

$$K' = b^{d-2} \left[K - \frac{N-2}{2} \int_{\Lambda/b}^{\Lambda} \frac{d^d \mathbf{q}}{(2\pi)^d} \frac{1}{q^2} \right] \qquad .$$

• After coarse-graining, renormalizing the fields, and rescaling,

$$\begin{aligned} K' &= b^{d-2} \zeta^2 \tilde{K} = b^{d-2} \left[1 - \frac{N-1}{K} I_d (b) \right] K \left[1 + \frac{N}{2K} I_d (b) \right] \\ &= b^{d-2} \left[K - \frac{N-2}{2} I_d (b) + \mathcal{O} (1/K) \right] , \end{aligned}$$

i.e., to lowest non-trivial order,

$$K' = b^{d-2}\left[K - \frac{N-2}{2}\int_{\Lambda/b}^{\Lambda}\frac{d^dq}{(2\pi)^d}\frac{1}{q^2}\right].$$

(i) Obtain the *differential* RG equation for $T = 1/K$, by considering $b = 1 + \delta\ell$. Sketch the flows for $d < 2$ and $d = 2$. For $d = 2 + \epsilon$, compute T_c and the critical exponent ν.

• Differential recursion relations are obtained for infinitesimal $b = 1 + \delta\ell$, as

$$K' = K + \frac{dK}{d\ell}\delta\ell = \left[1 + (d-2)\,\delta\ell\right]\left[K - \frac{N-2}{2}K_d\Lambda^{d-2}\delta\ell\right],$$

leading to

$$\frac{dK}{d\ell} = (d-2)\,K - \frac{N-2}{2}K_d\Lambda^{d-2}.$$

To obtain the corresponding equation for $T = 1/K$, we divide the above relation by $-K^2$, to get

$$\frac{dT}{d\ell} = (2-d)\,T + \frac{N-2}{2}K_d\Lambda^{d-2}T^2.$$

For $d < 2$, we have the two usual trivial fixed points: 0 (unstable) and ∞ (stable). The system is mapped unto higher temperatures by coarse-graining. The same applies for the case $d = 2$ and $N > 2$.

For $d > 2$, both 0 and ∞ are stable, and a non-trivial unstable fixed point appears at a finite temperature given by $dT/d\ell = 0$, *i.e.*

$$T^* = \frac{2\,(d-2)}{(N-2)\,K_d\Lambda^{d-2}} = \frac{4\pi\epsilon}{N-2} + \mathcal{O}\left(\epsilon^2\right).$$

In the vicinity of the fixed point, the flows are described by

$$\begin{aligned}\delta T' &= \left[1 + \frac{d}{dT}\left(\frac{dT}{d\ell}\right)\Big|_{T^*}\delta\ell\right]\delta T = \left\{1 + \left[(2-d) + (N-2)\,K_d\Lambda^{d-2}T^*\right]\delta\ell\right\}\delta T \\ &= (1 + \epsilon\delta\ell)\,\delta T.\end{aligned}$$

Thus, from

$$\delta T' = b^{y_T}\delta T = (1 + y_T\delta\ell)\,\delta T,$$

we get $y_T = \epsilon$, and

$$\nu = \frac{1}{\epsilon}.$$

(j) Consider a small symmetry breaking term $-h\int d^d\mathbf{x}\ \mathrm{tr}(M)$, added to the Hamiltonian. Find the renormalization of h, and identify the corresponding exponent y_h.

• As usual, h renormalizes according to

$$h' = b^d \zeta h = (1 + d\delta\ell)\left(1 - \frac{N-1}{2K} K_d \Lambda^{d-2} \delta\ell\right) h$$
$$= \left[1 + \left(d - \frac{N-1}{2K} K_d \Lambda^{d-2}\right)\delta\ell + \mathcal{O}\left(\delta\ell^2\right)\right] h.$$

From $h' = b^{y_h} h = (1 + y_h \delta\ell)\, h$, we obtain

$$y_h = d - \frac{N-1}{2K^*} K_d \Lambda^{d-2} = d - \frac{N-1}{N-2}(d-2) = 2 - \frac{\epsilon}{N-2} + \mathcal{O}\left(\epsilon^2\right).$$

Combining RG and symmetry arguments, it can be shown that the 3×3 matrix model is perturbatively equivalent to the $N = 4$ vector model at all orders. This would suggest that stacked triangular antiferromagnets provide a realization of the $\mathcal{O}(4)$ universality class; see P. Azaria, B. Delamotte, and T. Jolicoeur, J. Appl. Phys. **69**, 6170 (1991). However, non-perturbative (topological aspects) appear to remove this equivalence as discussed in S.V. Isakov, T. Senthil, Y.B. Kim, Phys. Rev. B **72**, 174417 (2005).

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3. *The roughening transition:* Consider a continuum interface model which in $d = 3$ is described by the Hamiltonian

$$\beta\mathcal{H}_0 = \frac{K}{2}\int d^2\mathbf{x}\,(\nabla h)^2 \quad ,$$

where $h(\mathbf{x})$ is the interface height at location $\mathbf{x}$. For a crystalline facet, the allowed values of h are multiples of the lattice spacing. In the continuum, this tendency for integer h can be mimicked by adding a term

$$-\beta U = y_0 \int d^2\mathbf{x}\, \cos(2\pi h),$$

to the Hamiltonian. Treat $-\beta U$ as a perturbation, and proceed to construct a renormalization group as follows:

(a) Show that

$$\left\langle \exp\left[i\sum_\alpha q_\alpha h(\mathbf{x}_\alpha)\right]\right\rangle_0 = \exp\left[\frac{1}{K}\sum_{\alpha<\beta} q_\alpha q_\beta C(\mathbf{x}_\alpha - \mathbf{x}_\beta)\right]$$

for $\sum_\alpha q_\alpha = 0$, and zero otherwise. ($C(\mathbf{x}) = \ln|\mathbf{x}|/2\pi$ is the Coulomb interaction in two dimensions.)

- The translational invariance of the Hamiltonian constrains $\left\langle \exp\left[i\sum_\alpha q_\alpha h\left(\mathbf{x}_\alpha\right)\right]\right\rangle_0$ to vanish unless $\sum_\alpha q_\alpha = 0$, as implied by the following relation

$$\exp\left(i\delta\sum_\alpha q_\alpha\right)\left\langle \exp\left[i\sum_\alpha q_\alpha h\left(\mathbf{x}_\alpha\right)\right]\right\rangle_0 = \left\langle \exp\left\{i\sum_\alpha q_\alpha\left[h\left(\mathbf{x}_\alpha\right)+\delta\right]\right\}\right\rangle_0$$
$$= \left\langle \exp\left[i\sum_\alpha q_\alpha h\left(\mathbf{x}_\alpha\right)\right]\right\rangle_0 .$$

The last equality follows from the symmetry $\mathcal{H}\left[h\left(\mathbf{x}\right)+\delta\right] = \mathcal{H}\left[h\left(\mathbf{x}\right)\right]$. Using general properties of Gaussian averages, we can set

$$\left\langle \exp\left[i\sum_\alpha q_\alpha h\left(\mathbf{x}_\alpha\right)\right]\right\rangle_0 = \exp\left[-\frac{1}{2}\sum_{\alpha\beta} q_\alpha q_\beta \left\langle h\left(\mathbf{x}_\alpha\right) h\left(\mathbf{x}_\beta\right)\right\rangle_0\right]$$
$$= \exp\left[\frac{1}{4}\sum_{\alpha\beta} q_\alpha q_\beta \left\langle \left(h\left(\mathbf{x}_\alpha\right) - h\left(\mathbf{x}_\beta\right)\right)^2\right\rangle_0\right].$$

Note that the quantity $\left\langle h\left(\mathbf{x}_\alpha\right) h\left(\mathbf{x}_\beta\right)\right\rangle_0$ is ambiguous because of the symmetry $h(\mathbf{x}) \to h(\mathbf{x})+\delta$. When $\sum_\alpha q_\alpha = 0$, we can replace this quantity in the above sum with the height difference $\left\langle \left(h\left(\mathbf{x}_\alpha\right) - h\left(\mathbf{x}_\beta\right)\right)^2\right\rangle_0$ which is independent of this symmetry. (The ambiguity, or symmetry, results from the kernel of the quadratic form having a zero eigenvalue, which means that inverting it requires care.) We can now proceed as usual, and

$$\left\langle \exp\left[i\sum_\alpha q_\alpha h\left(\mathbf{x}_\alpha\right)\right]\right\rangle_0 = \exp\left[\sum_{\alpha,\beta}\frac{q_\alpha q_\beta}{4}\int\frac{d^2q}{\left(2\pi\right)^2}\frac{\left(e^{i\mathbf{q}\cdot\mathbf{x}_\alpha} - e^{i\mathbf{q}\cdot\mathbf{x}_\beta}\right)\left(e^{-i\mathbf{q}\cdot\mathbf{x}_\alpha} - e^{-i\mathbf{q}\cdot\mathbf{x}_\beta}\right)}{Kq^2}\right]$$
$$= \exp\left[\sum_{\alpha<\beta} q_\alpha q_\beta \int\frac{d^2q}{\left(2\pi\right)^2}\frac{1-\cos\left(\mathbf{q}\cdot\left(\mathbf{x}_\alpha - \mathbf{x}_\beta\right)\right)}{Kq^2}\right]$$
$$= \exp\left[\frac{1}{K}\sum_{\alpha<\beta} q_\alpha q_\beta C\left(\mathbf{x}_\alpha - \mathbf{x}_\beta\right)\right],$$

where

$$C\left(\mathbf{x}\right) = \int\frac{d^2q}{\left(2\pi\right)^2}\frac{1-\cos\left(\mathbf{q}\cdot\mathbf{x}\right)}{q^2} = \frac{1}{2\pi}\ln\frac{|\mathbf{x}|}{a},$$

is the Coulomb interaction in two dimensions, with a short distance cutoff a.

(b) Prove that

$$\left\langle |h(\mathbf{x}) - h(\mathbf{y})|^2\right\rangle = -\left.\frac{d^2}{dk^2}G_k(\mathbf{x}-\mathbf{y})\right|_{k=0},$$

where $G_k(\mathbf{x}-\mathbf{y}) = \left\langle \exp\left[ik\big(h(\mathbf{x})-h(\mathbf{y})\big)\right]\right\rangle$.

• From the definition of $G_k\left(\mathbf{x}-\mathbf{y}\right)$,

$$\frac{d^2}{dk^2}G_k\left(\mathbf{x}-\mathbf{y}\right) = -\left\langle \left[h\left(\mathbf{x}\right)-h\left(\mathbf{y}\right)\right]^2 \exp\left[ik\left(h\left(\mathbf{x}\right)-h\left(\mathbf{y}\right)\right)\right]\right\rangle.$$

Setting k to zero results in the identity

$$\left\langle \left[h\left(\mathbf{x}\right)-h\left(\mathbf{y}\right)\right]^2\right\rangle = -\left.\frac{d^2}{dk^2}G_k\left(\mathbf{x}-\mathbf{y}\right)\right|_{k=0}.$$

(c) Use the results in (a) to calculate $G_k(\mathbf{x}-\mathbf{y})$ in perturbation theory to order of y_0^2. (Hint: Set $\cos(2\pi h) = \left(e^{2i\pi h}+e^{-2i\pi h}\right)/2$. The first order terms vanish according to the result in (a), while the second order contribution is identical in structure to that of the Coulomb gas described in this chapter.)

• Following the hint, we write the perturbation as

$$-U = y_0\int d^2x\cos\left(2\pi h\right) = \frac{y_0}{2}\int d^2x\left[e^{2i\pi h}+e^{-2i\pi h}\right].$$

The perturbation expansion for $G_k\left(\mathbf{x}-\mathbf{y}\right) = \left\langle\exp\left[ik\left(h\left(\mathbf{x}\right)-h\left(\mathbf{y}\right)\right)\right]\right\rangle \equiv \left\langle\mathcal{G}_k\left(\mathbf{x}-\mathbf{y}\right)\right\rangle$ is calculated as

$$\begin{aligned}\left\langle\mathcal{G}_k\right\rangle &= \left\langle\mathcal{G}_k\right\rangle_0 - \left(\left\langle\mathcal{G}_kU\right\rangle_0 - \left\langle\mathcal{G}_k\right\rangle_0\left\langle U\right\rangle_0\right)\\ &+\frac{1}{2}\left(\left\langle\mathcal{G}_kU^2\right\rangle_0 - 2\left\langle\mathcal{G}_kU\right\rangle_0\left\langle U\right\rangle_0 + 2\left\langle\mathcal{G}_k\right\rangle_0\left\langle U\right\rangle_0^2 - \left\langle\mathcal{G}_k\right\rangle_0\left\langle U^2\right\rangle_0\right) + \mathcal{O}\left(U^3\right).\end{aligned}$$

From part (a),

$$\left\langle U\right\rangle_0 = \left\langle\mathcal{G}_kU\right\rangle_0 = 0,$$

and

$$\left\langle\mathcal{G}_k\right\rangle_0 = \exp\left[-\frac{k^2}{K}C\left(\mathbf{x}-\mathbf{y}\right)\right] = \left(\frac{\left|\mathbf{x}-\mathbf{y}\right|}{a}\right)^{-\frac{k^2}{2\pi K}}.$$

Furthermore,

$$\begin{aligned}\left\langle U^2\right\rangle_0 &= \frac{y_0^2}{2}\int d^2\mathbf{x}'d^2\mathbf{x}''\left\langle\exp\left[2i\pi\left(h\left(\mathbf{x}'\right)-h\left(\mathbf{x}''\right)\right)\right]\right\rangle_0\\ &= \frac{y_0^2}{2}\int d^2\mathbf{x}'d^2\mathbf{x}''\left\langle\mathcal{G}_{2\pi}\left(\mathbf{x}'-\mathbf{x}''\right)\right\rangle_0 = \frac{y_0^2}{2}\int d^2\mathbf{x}'d^2\mathbf{x}''\exp\left[-\frac{\left(2\pi\right)^2}{K}C\left(\mathbf{x}'-\mathbf{x}''\right)\right],\end{aligned}$$

and similarly,

$$\begin{aligned}&\left\langle\exp\left[ik\left(h\left(\mathbf{x}\right)-h\left(\mathbf{y}\right)\right)\right]U^2\right\rangle_0 =\\ &= \frac{y_0^2}{2}\int d^2\mathbf{x}'d^2\mathbf{x}''\exp\Bigg\{-\frac{k^2}{K}C\left(\mathbf{x}-\mathbf{y}\right) - \frac{\left(2\pi\right)^2}{K}C\left(\mathbf{x}'-\mathbf{x}''\right)\\ &+\frac{2\pi k}{K}\left[C\left(\mathbf{x}-\mathbf{x}'\right)+C\left(\mathbf{y}-\mathbf{x}''\right)\right] - \frac{2\pi k}{K}\left[C\left(\mathbf{x}-\mathbf{x}''\right)+C\left(\mathbf{y}-\mathbf{x}'\right)\right]\Bigg\}.\end{aligned}$$

Thus, the second order part of $G_k(\mathbf{x}-\mathbf{y})$ is

$$\frac{y_0^2}{4}\exp\left[-\frac{k^2}{K}C(\mathbf{x}-\mathbf{y})\right]\int d^2\mathbf{x}'d^2\mathbf{x}''\exp\left[-\frac{4\pi^2}{K}C(\mathbf{x}'-\mathbf{x}'')\right]\cdot$$
$$\cdot\left\{\exp\left[\frac{2\pi k}{K}\left(C(\mathbf{x}-\mathbf{x}')+C(\mathbf{y}-\mathbf{x}'')-C(\mathbf{x}-\mathbf{x}'')-C(\mathbf{y}-\mathbf{x}')\right)\right]-1\right\},$$

and

$$G_k(\mathbf{x}-\mathbf{y})=e^{-\frac{k^2}{K}C(\mathbf{x}-\mathbf{y})}\left\{1+\frac{y_0^2}{4}\int d^2\mathbf{x}'d^2\mathbf{x}''e^{-\frac{4\pi^2}{K}C(\mathbf{x}'-\mathbf{x}'')}\left(e^{\frac{2\pi k}{K}\mathcal{D}}-1\right)+\mathcal{O}\left(y_0^4\right)\right\},$$

where

$$\mathcal{D}=C(\mathbf{x}-\mathbf{x}')+C(\mathbf{y}-\mathbf{x}'')-C(\mathbf{x}-\mathbf{x}'')-C(\mathbf{y}-\mathbf{x}').$$

(d) Write the perturbation result in terms of an effective interaction K, and show that perturbation theory fails for K larger than a critical K_c.

• The above expression for $G_k(\mathbf{x}-\mathbf{y})$ is very similar to that of obtained in dealing with the renormalization of the Coulomb gas of vortices in the XY model. Following the steps in the lecture notes, without further calculations, we find

$$\begin{aligned}G_k(\mathbf{x}-\mathbf{y})&=e^{-\frac{k^2}{K}C(\mathbf{x}-\mathbf{y})}\left\{1+\frac{y_0^2}{4}\times\frac{1}{2}\left(\frac{2\pi k}{K}\right)^2\times C(\mathbf{x}-\mathbf{y})\times 2\pi\int drr^3e^{-\frac{2\pi\ln(r/a)}{K}}\right\}\\&=e^{-\frac{k^2}{K}C(\mathbf{x}-\mathbf{y})}\left\{1+\frac{\pi^3k^2}{K^2}y_0^2C(\mathbf{x}-\mathbf{y})\int drr^3e^{-\frac{2\pi\ln(r/a)}{K}}\right\}.\end{aligned}$$

The second order term can be exponentiated to contribute to an effective coupling constant K_{eff}, according to

$$\frac{1}{K_{\text{eff}}}=\frac{1}{K}-\frac{\pi^3}{K^2}a^{2\pi/K}y_0^2\int_a^\infty drr^{3-2\pi/K}.$$

Clearly, the perturbation theory is inconsistent if the above integral diverges, *i.e.* if

$$K>\frac{\pi}{2}\equiv K_c.$$

(e) Recast the perturbation result in part (d) into renormalization group equations for K and y_0, by changing the "lattice spacing" from a to ae^{ℓ}.

• After dividing the integral into two parts, from a to ab and from ab to ∞, respectively, and rescaling the variable of integration in the second part, in order to retrieve the usual limits of integration, we have

$$\frac{1}{K_{\text{eff}}}=\frac{1}{K}-\frac{\pi^3}{K^2}a^{2\pi/K}y_0^2\int_a^{ab}drr^{3-2\pi/K}-\frac{\pi^3}{K^2}a^{2\pi/K}\times y_0^2b^{4-2\pi/K}\times\int_a^\infty drr^{3-2\pi/K}.$$

(To order y_0^2, we can indifferently write K or K' (defined below) in the last term.) In other words, the coarse-grained system is described by an interaction identical in form, but parameterized by the renormalized quantities

$$\frac{1}{K'} = \frac{1}{K} - \frac{\pi^3}{K^2} a^{2\pi/K} y_0^2 \int_a^{ab} dr r^{3-2\pi/K},$$

and

$$y_0'^{\,2} = b^{4-2\pi/K} y_0^2.$$

With $b = e^\ell \approx 1 + \ell$, these RG relations are written as the following differential equations, which describe the renormalization group flows

$$\begin{cases} \dfrac{dK}{d\ell} = \pi^3 a^4 y_0^2 + \mathcal{O}\left(y_0^4\right) \\ \dfrac{dy_0}{d\ell} = \left(2 - \dfrac{\pi}{K}\right) y_0 + \mathcal{O}\left(y_0^3\right) \end{cases}.$$

(f) Using the recursion relations, discuss the phase diagram and phases of this model.

• These RG equations are similar to those of the XY model, with K (here) playing the role of T in the Coulomb gas. For non-vanishing y_0, K is relevant, and thus flows to larger and larger values (outside of the perturbative domain) if y_0 is also relevant ($K > \pi/2$), suggesting a smooth phase at low temperatures ($T \sim K^{-1}$). At small values of K, y_0 is irrelevant, and the flows terminate on a fixed line with $y_0 = 0$ and $K \leq \pi/2$, corresponding to a rough phase at high temperatures.

(g) For large separations $|\mathbf{x} - \mathbf{y}|$, find the magnitude of the discontinuous jump in $\left\langle |h(\mathbf{x}) - h(\mathbf{y})|^2 \right\rangle$ at the transition.

• We want to calculate the long distance correlations in the vicinity of the transition. Equivalently, we can compute the coarse-grained correlations. If the system is prepared at $K = \pi/2^-$ and $y_0 \approx 0$, under coarse-graining, $K \to \pi/2^-$ and $y_0 \to 0$, resulting in

$$G_k\left(\mathbf{x} - \mathbf{y}\right) \to \left\langle \mathcal{G}_k \right\rangle_0 = \exp\left[-\frac{2k^2}{\pi} C\left(\mathbf{x} - \mathbf{y}\right)\right].$$

From part (b),

$$\left\langle \left[h\left(\mathbf{x}\right) - h\left(\mathbf{y}\right)\right]^2 \right\rangle = -\left.\frac{d^2}{dk^2} G_k\left(\mathbf{x} - \mathbf{y}\right)\right|_{k=0} = \frac{4}{\pi} C\left(\mathbf{x} - \mathbf{y}\right) = \frac{2}{\pi^2} \ln\left|\mathbf{x} - \mathbf{y}\right|.$$

On the other hand, if the system is prepared at $K = \pi/2^+$, then $K \to \infty$ under the RG (assuming that the relevance of K holds also away from the perturbative regime), and

$$\left\langle \left[h\left(\mathbf{x}\right) - h\left(\mathbf{y}\right)\right]^2 \right\rangle \to 0.$$

Thus, the magnitude of the jump in $\left\langle [h(\mathbf{x}) - h(\mathbf{y})]^2 \right\rangle$ at the transition is

$$\frac{2}{\pi^2} \ln |\mathbf{x} - \mathbf{y}| .$$

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4. *Roughening and duality:* Consider a discretized version of the Hamiltonian in the previous problem, in which for each site i of a square lattice there is an integer valued height h_i. The Hamiltonian is

$$\beta\mathcal{H} = \frac{K}{2} \sum_{<i,j>} |h_i - h_j|^\infty \quad ,$$

where the "∞" power means that there is no energy cost for $\Delta h = 0$; an energy cost of $K/2$ for $\Delta h = \pm 1$; and $\Delta h = \pm 2$ or higher *are not allowed* for neighboring sites. (This is known as the restricted solid on solid (RSOS) model.)

(a) Construct the dual model either diagrammatically, or by following these steps:

(i) Change from the N site variables h_i, to the $2N$ bond variables $n_{ij} = h_i - h_j$. Show that the sum of n_{ij} around any plaquette is constrained to be zero.

(ii) Impose the constraints by using the identity $\int_0^{2\pi} d\theta e^{i\theta n}/2\pi = \delta_{n,0}$, for integer n.

(iii) After imposing the constraints, you can sum freely over the bond variables n_{ij} to obtain a dual interaction $\tilde{v}(\theta_i - \theta_j)$ between dual variables θ_i on neighboring plaquettes.

• (i) In terms of bond variables $n_{ij} = h_i - h_j$, the Hamiltonian is written as

$$-\beta\mathcal{H} = -\frac{K}{2} \sum_{\langle ij \rangle} |n_{ij}|^\infty .$$

Clearly,

$$\sum_{\text{any closed loop}} n_{ij} = h_{i_1} - h_{i_2} + h_{i_2} - h_{i_3} + \cdots + h_{i_{n-1}} - h_{i_n} = 0,$$

since $h_{i_1} = h_{i_n}$ for a closed path.

(ii) This constraint, applied to the N plaquettes, reduces the number of degrees of freedom from an apparent $2N$ (bonds), to the correct figure N, and the partition function becomes

$$Z = \sum_{\{n_{ij}\}} e^{-\beta\mathcal{H}} \prod_\alpha \delta_{\sum_{\langle ij \rangle} n_{ij}^\alpha, 0},$$

where the index α labels the N plaquettes, and n_{ij}^α is non-zero and equal to n_{ij} only if the bond $\langle ij \rangle$ belongs to plaquette α. Expressing the Kronecker delta in its exponential representation, we get

$$Z = \sum_{\{n_{ij}\}} e^{-\frac{K}{2} \sum_{\langle ij \rangle} |n_{ij}|^\infty} \prod_\alpha \left(\int_0^{2\pi} \frac{d\theta_\alpha}{2\pi} e^{i\theta_\alpha \sum_{\langle ij \rangle} n_{ij}^\alpha} \right) .$$

(iii) As each bond belongs to two neighboring plaquettes, we can label the bonds by $\alpha\beta$ rather than ij, leading to

$$Z = \left(\prod_\gamma \int_0^{2\pi} \frac{d\theta_\gamma}{2\pi}\right) \sum_{\{n_{\alpha\beta}\}} \exp\left(\sum_{\langle\alpha\beta\rangle} \left\{-\frac{K}{2}\left|n_{\alpha\beta}\right|^\infty + i\left(\theta_\alpha - \theta_\beta\right) n_{\alpha\beta}\right\}\right)$$
$$= \left(\prod_\gamma \int_0^{2\pi} \frac{d\theta_\gamma}{2\pi}\right) \prod_{\langle\alpha\beta\rangle} \sum_{n_{\alpha\beta}} \exp\left(\left\{-\frac{K}{2}\left|n_{\alpha\beta}\right|^\infty + i\left(\theta_\alpha - \theta_\beta\right) n_{\alpha\beta}\right\}\right).$$

Note that if all plaquettes are traversed in the same sense, the variable $n_{\alpha\beta}$ occurs in opposite senses (with opposite signs) for the constraint variables θ_α and θ_β on neighboring plaquettes. We can now sum freely over the bond variables, and since

$$\sum_{n=0,+1,-1} \exp\left(-\frac{K}{2}|n| + i\left(\theta_\alpha - \theta_\beta\right) n\right) = 1 + 2e^{-\frac{K}{2}} \cos\left(\theta_\alpha - \theta_\beta\right),$$

we obtain

$$Z = \left(\prod_\gamma \int_0^{2\pi} \frac{d\theta_\gamma}{2\pi}\right) \exp\left(\sum_{\langle\alpha\beta\rangle} \ln\left[1 + 2e^{-\frac{K}{2}} \cos\left(\theta_\alpha - \theta_\beta\right)\right]\right).$$

(b) Show that for large K, the dual problem is just the XY model. Is this conclusion consistent with the renormalization group results of the previous problem? (Also note the connection with the loop model.)

• This is the loop gas model , and for K large,

$$\ln\left[1 + 2e^{-\frac{K}{2}} \cos\left(\theta_\alpha - \theta_\beta\right)\right] \approx 2e^{-\frac{K}{2}} \cos\left(\theta_\alpha - \theta_\beta\right),$$

and

$$Z = \left(\prod_\gamma \int_0^{2\pi} \frac{d\theta_\gamma}{2\pi}\right) e^{\sum_{\langle\alpha\beta\rangle} 2e^{-\frac{K}{2}} \cos(\theta_\alpha - \theta_\beta)}.$$

This is none other than the partition function for the XY model, if we identify

$$K_{\rm XY} = 4e^{-\frac{K}{2}},$$

consistent with the results of another problem, in which we found that the low temperature behavior in the roughening problem corresponds to the high temperature phase in the XY model, and vice versa.

(c) Does the one dimensional version of this Hamiltonian, i.e. a 2d interface with

$$-\beta\mathcal{H} = -\frac{K}{2}\sum_i \left|h_i - h_{i+1}\right|^\infty \quad ,$$

have a roughening transition?
• In one dimension, we can directly sum the partition function, as

$$
\begin{aligned}
Z &= \sum_{\{h_i\}} \exp\left(-\frac{K}{2}\sum_i |h_i - h_{i+1}|^\infty\right) = \sum_{\{n_i\}} \exp\left(-\frac{K}{2}\sum_i |n_i|^\infty\right) \\
&= \prod_i \sum_{n_i} \exp\left(-\frac{K}{2}|n_i|^\infty\right) = \prod_i \left(1 + 2e^{-K/2}\right) = \left(1 + 2e^{-K/2}\right)^N ,
\end{aligned}
$$

($n_i = h_i - h_{i+1}$). The expression thus obtained is an analytic function of K (for $0 < K < \infty$), in the $N \to \infty$ limit, and there is therefore no phase transition at a finite non-zero temperature.

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