Hysteresis on Concurrent Synchronization of Neural Oscillators and Multiple Local Minima in Effective Free Energy

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Concurrent synchronization of neural oscillators connected with diffusive couplings can explain various computational and perceptional mechanism in the brain. Especially, there have been significant developments on the theory of neural oscillator dynamics toward time-invariant coupling configuration. We study the neural oscillator dynamics with time-variant diffusive couplings and find a hysteresis which may explain the effect of short memory. The existence of hysteresis implies co-existence of multiple stable states which is path-dependent. We introduce an effective free energy where entropic contribution comes from the order gained through synchronization of neurons and show that concurrent synchronization corresponds to the co-existence of multiple local minima in the effective free energy.

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INTRODUCTION

Concurrent synchronization is a regime where the whole system can be divided into multiple groups of fully synchronized elements while dynamics of each group remains independent [3]. In the brain, concurrent synchronization phenomena are known to occur at various scales, and are heavily studied at both experimental and computational levels. In particular, synchronization phenomena in neural oscillator network [1, 2] has been gathering significant attentions as an underlying mechanism for visual grouping problem [4, 5], neural computation [6], neural communications [7] and certain types of phase transitions in physics [8].

Synchronizations in a network of neurons are discussed by connecting each neural oscillators with diffusive couplings whose coupling strengths are controlled through external current inputs [9, 10]. Different input signals are assigned on each neurons corresponding to the "color" or "intensity" in each "pixel" in order to simulate the perception of an image in the brain. The algorithm using this mechanism is also shown to achieve promising result in visual grouping problem [4].

It is possible to account for the effect of memory and learning by changing the coupling geometry or strength in neural network where the system fall into the memorized state more likely under predetermined coupling [13, 14]. There is also an alternative approach to explain memory and learning where we simulate the given picture in the first layer of neural network while the second layer is used to represent the effect of memory by coupling neurons in first layer and second layer. Even though there have been many developments on the properties of neural oscillator network, discussions are mostly concentrated on the final state into which the system contract with diffusive couplings. However, in actual perceptional process, it is natural to consider time-variant dynamics which changes coupling strength according to time. Specifically, it is more realistic to consider the model where the brain "gradually" perceives the image rather than understands every detail of the image at once.

In this paper, we study the dynamical contraction of neural oscillator network with diffusive couplings where overall coupling strength changes in a time-variant way. When increasing and decreasing coupling strength adiabatically, we find a hysteresis effect in which we observe different contraction phases even for the same coupling strength, depending on the "path" the system followed. The hysteresis we observed may provide a starting point for the search of memory whose correlation time is relatively short compared to the memory which is simulated by assigning an ideal coupling configurations [14].

The existence of hysteresis implies that the coexistence of several stable states into which the system can contract [15]. Therefore, we are tempted to search for a virtual energy function which has several local minima to explain the nature of hysteresis. We introduce an effective free energy for neural oscillator network where the entropic contributions comes from the order gained through synchronization of neural oscillators while the internal energy can be quantified as the cost of diffusive coupling. Using this effective free energy description, we explain a peculiar synchronization phenomena observed for static coupling strength case. Finally, we discuss a possible crossover between physical system and neural oscillator networks.

THE MODEL

FitzHugh-Nagumo neural oscillator: Consider a network of neural oscillators defined on a square lattice which perceive an image and whose dynamics are dependent on the inputs on each corresponding "pixel" (Fig. 1). Neural oscillator consists of two state vector (v, w) where v is the membrane potential of the oscillator while w is an internal state variable representing gate voltage.

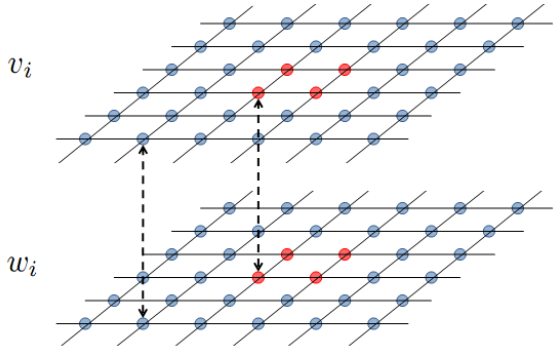

FIG. 1: Neural oscillators are defined on square lattice and membrane potential v is connected through diffusive couplings while gate voltage w is decoupled each other. The dynamics governing each oscillator is controlled by input on the pixel corresponding to each oscillator reflecting the "color" of the image.

We use a modified form of FitzHugh-Nagumo neural oscillators to mimic neural dynamics [1, 2],

$$\dot{v_i} = 3v_i - v_i^3 - v_i^7 + 2 - w_i + s_i \tag{1}$$

$$\dot{w_i} = c[\alpha(1 + \tanh(\beta v_i)) - w_i] \tag{2}$$

where s_i represents the external current input, and α, β and c are strictly positive constants. The external current input s_i represents the "color" or "intensity" imposed on each pixel. The two state vectors undergoes oscillatory behavior whose frequencies are determined by the intensity of s_i. We show the typical behavior of FitzHugh-Nagumo oscillator for different input s_i where frequencies of spikes increas as s_i increases (Fig. 2). Phenomenologically, we can see that v makes a spiking oscillations when it exceeds a certain threshold value around $v = -1$.

Here, we connect only membrane potential v_i through diffusive couplings while w_i are left uncoupled. The entire oscillatory dynamics are described by

$$\dot{\mathbf{x}}_i = \mathbf{f}_i(\mathbf{x}_i, s_i, t) - k\sum_{i \neq j} D_{ij}(v_i - v_j) \tag{3}$$

where $\mathbf{f}_i(\mathbf{x}_i, s_i, t)$ denotes the dynamics of uncoupled neural oscillators and D_{ij} denotes diffusive coupling strength. The state vectors of the oscillators are denoted by $\mathbf{x}_i = [v_i, w_i]^T$ and k is strictly positive constant representing a tuning parameter for overall interactions. Diffusion coupling is implemented commonly by the Gaussian coupling [4]

$$D_{ij} = \exp\left(-\frac{|s_i - s_j|^2}{\beta}\right) \quad (i \neq j) \tag{4}$$

$$d(i, j) \leq r \tag{5}$$

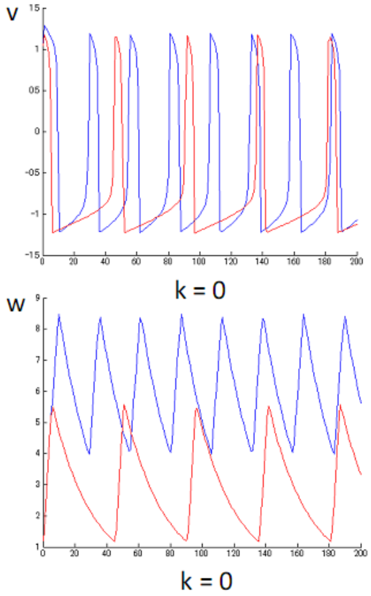

FIG. 2: The fast oscillatory mode colored with red is the behavior of neural oscillator with $s_i = 4$ while the slow oscillatory behavior with blue corresponds to neural oscillator with $s_i = 1$. Other parameters are taken as $c = 0.04$, $\alpha = 12$ and $\beta = 1$.

where $d(i, j)$ is the distance between two neural oscillators which is defined as $|a - b| + |c - d|$ for two neurons characterized by the positions in the square lattice with (a, b) and (c, d). For $d(i, j) > r$, we define $D_{ij} = 0$. Therefore, we cut the diffusive coupling by introducing the effective length r which characterizes the "focus" of perception and $r = 1$ represent nearest neighboring couplings.

In our model, the Jacobian matrix of the dynamics can be written as

$$\mathbf{J} = \frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{s}, t)}{\partial \mathbf{x}} + k\mathbf{D} \tag{6}$$

where

$$\mathbf{D}_{ii} = -\sum_{j \neq i} \mathbf{D}_{ij}. \tag{7}$$

We note that for the system consisting of the same s_i for all neural oscillators, off-diagonal element of $\mathbf{D}$ is identical to the Hamiltonian in tight-binding model when $r = 1$. The diffusion matrix $\mathbf{D}$ is nonzero only for membrane potential v_i.

Property of Diffusion Matrix: Essentially speaking, the oscillatory mode corresponding to the eigenvector $\mathbf{x}_i$ of Jacobian matrix with

$$\mathbf{J}\mathbf{x}_i = \lambda_i \mathbf{x}_i \tag{8}$$

undergoes the dynamics of $\exp(\lambda_i t)$ for a short time of period [10, 12]. Therefore, positive λ_i describes diverging evolution while negative λ_i describes converging decay. If $\mathbf{J}$ is negative definite for all the time evolution, the system will fall into a single trajectory through exponential decay.

Similarly we can discuss the contraction into a subspace in $\mathbb{R}^N$ by considering projection matrices U and V representing the projection into two mutually orthogonal linear subspace

$$\mathbf{U}\mathbf{U}^T + \mathbf{V}\mathbf{V}^T = I \tag{9}$$

where I is $N \times N$ identity matrix. If the Jacobian matrix $\mathbf{J}$ satisfies

$$\mathbf{x}\mathbf{U}\mathbf{J}\mathbf{U}^T\mathbf{x}^T < 0, \ \ (\mathbf{x} \neq 0) \tag{10}$$

or $\mathbf{U}\mathbf{J}\mathbf{U}^T$ is negative definite, the system will eventually fall into the subspace characterized by V.

For the general property of diffusion matrix $\mathbf{D}$, we have the following theorem.

Theorem 1. *Let us assume that neural oscillators are sitting on the edge on a connected graph $G = (V, E)$. Diffusion matrix $\boldsymbol{D}$ connects neural oscillators according to the vertex of the graph with*

$$\boldsymbol{D}_{ij} > 0, \ \ (i, j) \in V \tag{11}$$

$$\boldsymbol{D}_{ij} = 0, \ \ (i, j) \notin V \tag{12}$$

$$\boldsymbol{D}_{ii} = -\sum_{j \neq i} \boldsymbol{D}_{ij}. \tag{13}$$

Then, for large enough k, the system contract into the linear subspace characterized by $\boldsymbol{v} = \boldsymbol{1}$.

For the proof of the theorem, we need the following lemma.

Lemma 1. *All the eigenvalues of $\boldsymbol{D}$ is strictly negative except that the eigenvalue for $\boldsymbol{v} = \boldsymbol{1}$ is zero.*

Proof. Let us define matrices $N \times N$ matrix $\mathbf{I}^{ab}$ which acting only on v_i where

$$\mathbf{I}^{ab}(i, j) = -1 \qquad (i, j) = (a, a), (b, b) \tag{14}$$

$$\mathbf{I}^{ab}(i, j) = 1 \qquad (i, j) = (a, b), (b, a) \tag{15}$$

$$\mathbf{I}^{ab}(i, j) = 0 \qquad \text{otherwise.} \tag{16}$$

Therefore, (i, j) represents the indexes of the matrix I^{AB}. For example, I^{12} is defined as

$$I^{12} = \begin{pmatrix} -1 & 1 & 0 & \dots \\ 1 & -1 & 0 & \dots \\ 0 & 0 & 0 & \\ \vdots & \vdots & & \ddots \end{pmatrix} \tag{17}$$

The inner product of ^{a,b} with $\mathbf{x}$ is

$$\mathbf{x}\mathbf{I}(a, b)\mathbf{x}^T = -(x_a - x_b)^2 \leq 0. \tag{18}$$

which is always negative or zero. Since the matrix $\mathbf{D}$ can be written as

$$\mathbf{D} = \sum_{a \neq b} \mathbf{I}^{ab} D_{ab} \tag{19}$$

with respect to the matrix $\mathbf{I}(a, b)$ where $D_{ab} \geq 0$, we can calculate the inner product of $\mathbf{D}$ as

$$\mathbf{x}\mathbf{D}\mathbf{x}^T = -\sum_{i \neq j} D_{ij}(x_i - x_j)^2. \tag{20}$$

From the above expression, we can easily see that

$$\mathbf{x}\mathbf{D}\mathbf{x}^T \leq 0 \tag{21}$$

and $\mathbf{D}$ is negative semi-definite.

Since the graph G is connected, there is no neural oscillator which is completely isolated from other neural oscillators. Therefore, for given integer i, there exists $j (\neq i)$ such that $D_{ij} > 0$ meaning that neural oscillators for i and j are connected through diffusive coupling meaning that $(i, j) \in V$. Therefore, in order for $\mathbf{x}\mathbf{D}\mathbf{x}^T = 0$, we must have $x_i = x_j$. Applying the same discussions for all integer i, we end up with $x_i = x_j$ for all i and j for a connected graph G. □

Then the proof of the theorem is as follows.

Proof. If we take V as the projection into the space $\mathbf{v} = \mathbf{1}$ and U as the projection into the orthogonal space with respect to $\mathbf{v} = \mathbf{1}$, we have

$$\mathbf{x}\mathbf{U}\mathbf{J}\mathbf{U}^T\mathbf{x}^T < 0, \ \ (\mathbf{x} \neq 0) \tag{22}$$

for large enough k since the individual dynamics of $\partial \mathbf{f}(\mathbf{x}, \mathbf{s}, t)/\partial \mathbf{x}$ is bounded by constant. Therefore, the system will eventually fall into the regime where all the oscillators are synchronized. □

Two Neural Oscillator: At first, let us observe the oscillatory behavior of two neural oscillators coupled with diffusive coupling. From now on, our discussions on synchronization will be concentrated on the oscillatory behavior of v_i since v_i and w_i are strongly correlated and when v_i are synchronized, we can safely say that w_i are synchronized in the same manner. From the discussion in the last section, we expect that for large k the system will eventually synchronized into the mode characterized by $(v_1, v_2) = (1, 1)$ [Fig. 3b]. However at an intermediate coupling, we observe that two period of red neural oscillator coincides with one period of blue neural oscillator and partially synchronized each other (Fig. 3a). We note that originally the ratio of frequency was around 1.7 which is smaller than 2.

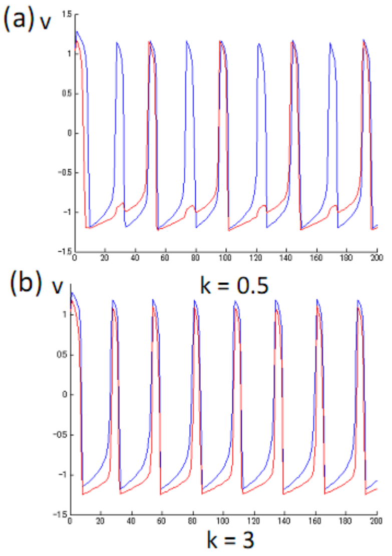

FIG. 3: While (b) describes the perfect synchronization between blue and red neural oscillators, (a) exhibit concurrent synchronization where we observe two types of oscillatory behavior.

We can understand this synchronization as follows. For red neural oscillator, the duration between two successive spikes is shorter than the one for blur neural oscillators. Between spiking oscillations, neural oscillators undergo dissipations from the diffusion matrix **D** and lose its amplitude. If v_i cannot exceed threshold, the associated spiking oscillation cannot happen. At intermediate coupling strength, red neural oscillators can survive the dissipative phase between spikes while blue neural oscillator cannot survive the dissipations. As a result, we observe two types of synchronization involving only red neuron or involving both red and blue neurons.

Therefore, neural oscillators undergo two phases of dynamics corresponding dissipation and recovery. Between spikes, dissipative effect totally dominates the dynamics while during spiking oscillation the amplitude is recovered until a certain upper limit around $v_i = 1$.

EFFECTIVE FREE ENERGY

Checker Board Picture: Next, let us consider a picture of red and blue points defined in an alternating way like a checker board (Fig. 4). We define diffusive couplings with $r = 1$, meaning that only neighboring red and blue neurons are coupled. The off-diagonal elements of diffusive matrix **D** is the same as the Hamiltonian for 2 dimensional tight-binding model. Our motivation for choosing checker board picture is as follows. Because of the symmetry in **D** with respect to the exchange of red and blue neurons, it is impossible to impose diffusive couplings based on memory a priori for this model. Therefore, this model provides us an ideal testbed for the study of memory effect without predetermined coupling configurations. For the simulation, we use randomly generated neural oscillators whose spiking timings are at random without diffusive couplings.

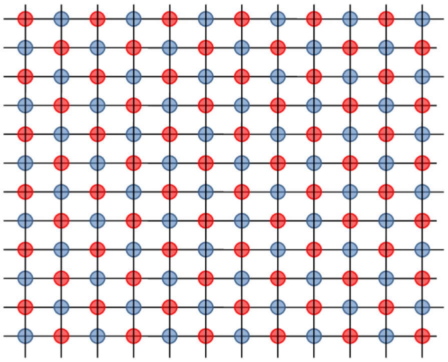

FIG. 4: Neurons are given inputs whose pattern is alternating like a checker board.

For $k = 0$, oscillators are totally desynchronized each other and there is no order in the system where all the spikes arrive in a random way. For $k = 0.06$, we can observe that only the red neural oscillators are synchronized while blue neural oscillators are totally desynchronized (Fig. 5a). Therefore, the brain recognizes red neural oscillators as one group while it does not recognize blue neural oscillators.

On the other hand, for $k = 0.07$ (Fig. 5b), blue neurons seem to acquire synchronized dynamics while red neurons are synchronized. However, we need to be careful since the period of oscillation for red neural oscillators is twice as the one for blue neural oscillators. In fact, we observe two types of synchronization for blue neural oscillators. Around the half of blue neurons are synchronized with $2n$-th spike of red neurons while other half are synchronized with $2n + 1$-th spike of red neurons. Therefore, the brain recognizes red neurons as one group and try to associate them with blue neurons, but still blue neurons are not perceived as one group. In fact, this synchronization is quite surprising since there is no direct coupling between red neurons. Therefore, red neurons gain synchronization through blue neurons which are still desynchronized.

If we increase the value of k furthermore to $k = 0.1$ (Fig. 5c), we observe the synchronization of red and blue neurons where the ratio of frequency is 2 and there are two types of oscillators surviving in this phase where neurons recognize red and blue inputs as one group or only red inputs as one group. Therefore the brain has two choices of judging whether the system as a whole picture of one color, or the picture of alternating colors. Finally, if we increase the value of k to $k = 1$, all the neurons are perfectly synchronized (Fig. 5d).

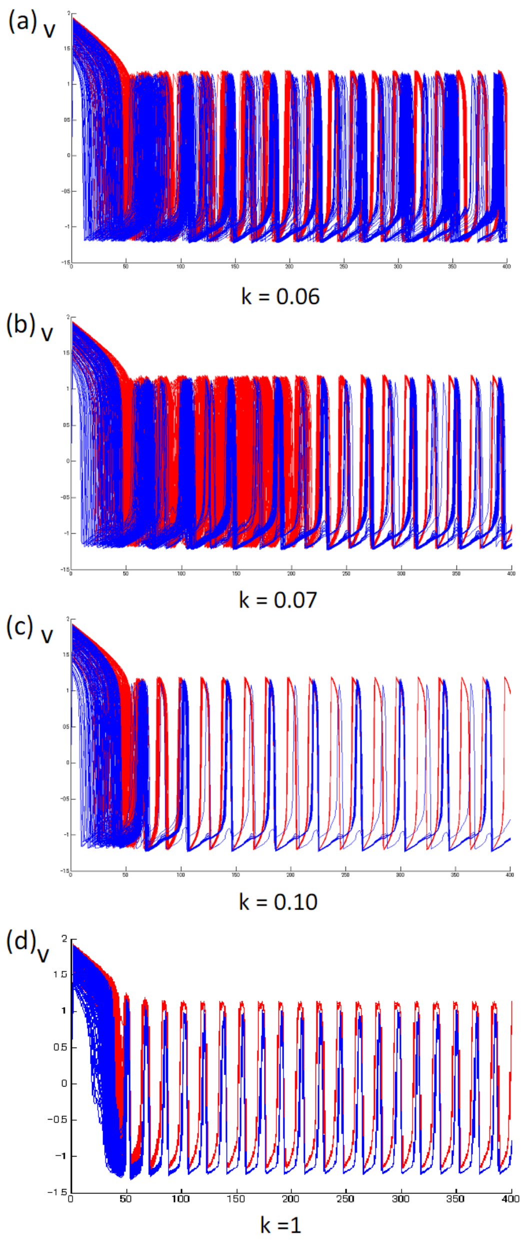

FIG. 5: (a) Only red neurons are synchronized. (b) Blue neurons are synchronized with red neurons, but with different spikes. Therefore blue neurons are still not perfectly synchronized. (c) Red and blue neurons are synchronized where the ratio of frequencies is 2. (d) All the neurons are perfectly synchronized.

Entropy and Synchronization: We can explain the contraction into above concurrent synchronization regimes by introducing effective free energy of the system with the form $F = U - TS$. Ar first, the system is totally desynchronized without any order which implies large entropy with $S = N$. On the other hand, in fully synchronized mode, the system is perfectly in order which implies decrease of entropy with $S = 0$. Therefore, the entropy of the system can be represented simply as

$$S = N - N_0 \tag{23}$$

where N_0 is the number of synchronized neurons. For example, for the system where only the red neurons are synchronized, we have $S = N/2$.

In FitzHugh-Nagumo neural oscillator, large s_i means frequent spiking oscillation and the effect of dissipation tends to be smaller due to frequent recovery process. Since this can be viewed as less frequent exposure to diffusions, we can assign effective temperatures to each neuron as $T(s_i)$ which is decreasing function of s_i reflecting the oscillatory frequency. Therefore we can regard each neuron as oscillator held in contact with the thermal bath of $T(s_i)$.

The total entropic contribution for free energy $F = U - TS$ can be defined as

$$\sum_i T(s_i)\delta_i \tag{24}$$

where $\delta_i = 1$ for synchronized oscillator and otherwise $\delta_i = 0$. In our two color problems on checker board pattern, it can be written as

$$T_r S_r + T_b S_b \tag{25}$$

where $T_{r,b}$ are temperature for red and blue oscillators where $T_r < T_b$, while $S_{r,b}$ represents how much of oscillators of each color have been synchronized.

Next, we quantify the cost of diffusive couplings for each oscillatory mode. Since $\mathbf{D}$ describes dissipations, the "internal energy" can be assessed as an expectation value of decay rate

$$U = -k\mathbf{x}\mathbf{D}\mathbf{x} \tag{26}$$

which where $\mathbf{x}$ is normalized so that the norm is unity. For example, for the trivial oscillatory mode $\mathbf{x} = \mathbf{1}$, the internal energy becomes minimum with $U = 0$. We emphasize that this internal energy is effective one and different from actual energy of oscillators since neurons keep their oscillations even with $U = 0$. We also note that $-\mathbf{D}$ can be viewed as complex value Hamiltonian of the system under dynamical evolution.

By combining them, we can represent the free energy of the system as

$$F = U - T_r S_r - T_b S_b. \tag{27}$$

Therefore, the problem can be reduced into the tight binding model where each oscillator is coupled to thermal bath of different temperature.

Let us apply this effective free energy to our model. There are four phases of synchronizations in this model. At first, the random phase describes the phase where no

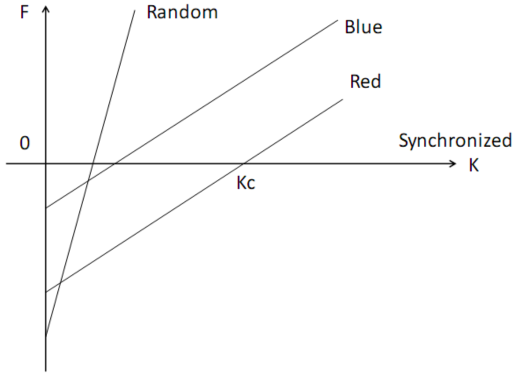

FIG. 6: The plot of free energy with respect to coupling strength k.

oscillators are synchronized and whose free energy is

$$F_0 = kU_0. \tag{28}$$

Next, the phase of partial synchronizations where only red or blue neurons are synchronized has the ferr energy

$$F_1 = kU_1 - T_b N/2 \tag{29}$$
$$F_2 = kU_1 - T_r N/2 \tag{30}$$

where $U_0 > U_1$. Finally, perfectly synchronized phase has the free energy

$$F_3 = -T_r N/2 - T_b N/2. \tag{31}$$

We plot free energy F with respect to k (Fig 6). One can see that at first synchronization with respect to red neurons will be observed. Around $k = k_c$, we can observe the co-existence of two phases of red synchronization and perfect synchronization where the oscillatory modes corresponding to two local minima of free energy are co-existing each other. Therefore, we can view the concurrent synchronization as the co-existence of several local minima of free energy.

HYSTERESIS

Since the effective energy possesses multiple local minima, it is natural to make a conjecture that there will be a hysteresis depending on the path which the system follows. Let us discuss how the dynamics of neural oscillator network changes when the coupling strength k is adiabatically changed. At first, we adiabatically increase the coupling strength from 0.3 to 10 (Fig. 7).

For small k, red neurons are synchronized at first and then, blue neural oscillators are synchronized. Finally, all the neural oscillators are synchronized in such a way that the ratio of frequency is 2 (Fig. 8). Surprisingly, for large value of k such as $k = 10$, we still have the synchronization with the ratio 2 rather than perfectly synchronized phase. This fact is sharply in contrast with that for $k = 1$ the system was perfectly synchronized in the static coupling strength case. This observation clearly shows the co-existence of two local minima of free energy whose energy barrier is large enough to separate each phases toward adiabatic increase of coupling strength.

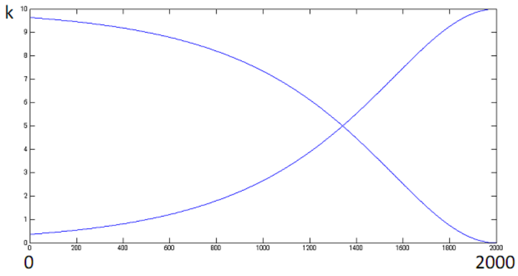

FIG. 7: We describe the adiabatic increase and decrease of the coupling strength k implemented through cosh function.

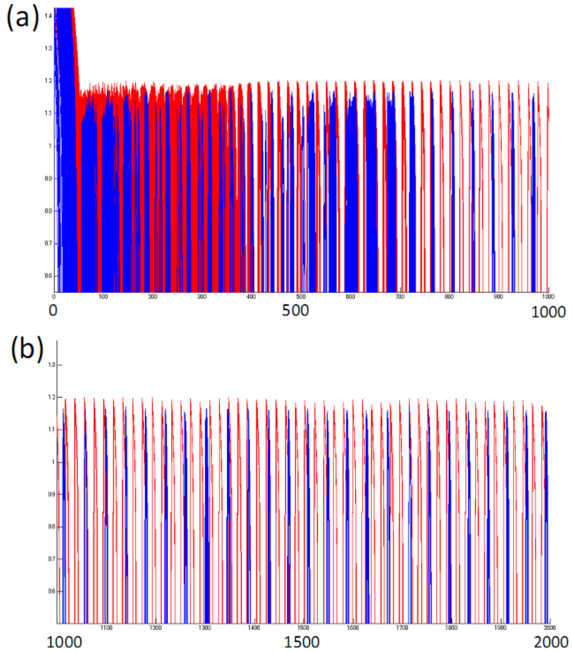

FIG. 8: Increasing k for (a) $0 < t < 1000$ (b) $1000 < t < 2000$.

Next, we decrease the coupling strength from 9.7 to 0 (Fig. 9) adiabatically. Soon after we switch on the coupling, the system fall into the regime where all the neurons are synchronized perfectly. However, this perfect synchronization is rather stable even a very small values of $k \sim 0$. Therefore, we observe totally different phases even for the same k when we increase and decrease the coupling strength.

In the context of perceptual visual grouping problem, this hysteresis effect can be viewed as a manifestation of

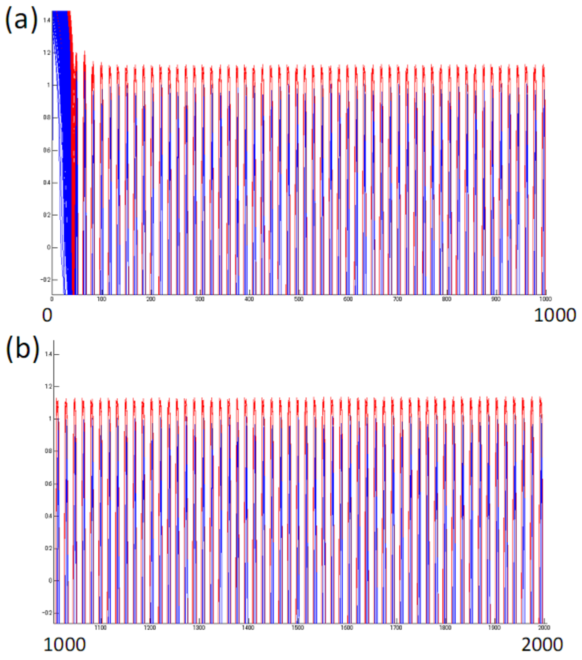

FIG. 9: Decreasing k for (a) $0 < t < 1000$ (b) $1000 < t < 2000$.

short time memory. For example, once a certain synchronization is established with very large k, the neural oscillators can easily caught the same synchronization even when we decrease the value of k. Therefore, the brain is affected by what we saw for a short while ago.

DISCUSSIONS AND SUMMARY

The reason for the existence of hysteresis is due to the quantized oscillations of neural oscillators and therefore, it is rather hard to apply classical description such as free energy to the study of these phenomena. However, we emphasize that there is a similarity between our observations and optical bi-stability where two resonant transmissions states are possible and stable, dependent on the input [15]. Since well defined oscillator spikes can be viewed as optical mode in cavity, cavity photon system may be used for the study of neural oscillator networks and vice versa. Also, since concurrent synchronization possesses multiple stable oscillatory modes, we can consider to apply neural oscillator network for optimal solution search problem. Especially, since the system can remain in several stable states simultaneously, neural oscillator network can search the solution in a parallel way. We note that usual physical system does not possess such parallel nature since the system will fall into a single state after infinitely long time of dissipations.

In summary, we studied the oscillatory behavior of neurons toward time-variant diffusive couplings where we find hysteresis effect. We explained the possible types of synchronization regimes through thermodynamical argument. Time varying dynamics of coupled neural oscillators may explain the staggered memory in the perceptions and provide us a starting point for further discussions on dynamical evolutions of neural network.

Footnote: The paper is an original work developed by B.Y on his own. B.Y. thank Megumi Matsutani for explaining neural network and nonlinear control theory.

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