Electrostatics of DNA in Solutes and Liouville Theory

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There are many fascinating aspects of the behavior of DNA inside solutes. In particular, same charge DNA have been observed to attract under a wide range of solution conditions. We review statistical models constructed to account for this behavior and point out its similarities to Liouville theory–a theory of two-dimensional quantum gravity. Inspired by this, we approximate the theory by a deformed 'time-like Liouville theory' and examine the interactions in the UV and IR. Surprisingly this model exhibits some nice qualitative features, which we describe.

INTRODUCTION

DNA molecules exhibit non-trivial behavior in various solute conditions that are often counter intuitive[1]. Although they do not exhibit any non-intuitive behavior under physiological conditions, when polyvalent cations are added to the solution some surprising phases can be observed. For example, they can form toroidal condensates, or form bilayer complexes with charged lipids. Most of the exotic behavior is attributed to the fact that under certain conditions of the solution, the electrostatics become highly non-trivial. The most surprising observation is that like-charged DNA molecules interact at short range.

Electrostatics of solutes is an extremely well studied topic. In the zeroth order approximation, we may take the mean field theory approach which boils down to solving the Poisson-Boltzmann equation. Given ion species $i = 1, \cdots, n$ with charge z_i and number density n_i, we may approximate n_i and the electrostatic potential Ψ to be

$$\Box\Psi = -\frac{4\pi}{\epsilon}(\rho_E + \sum_i z_i e n_i) \qquad (1)$$

$$n_i = n_i^0 \exp(-z_i e\Psi/kT) \qquad (2)$$

in the presence of an external source with charge density ρ_E. This classical problem in and of itself is highly non-trivial and is the subject of active research both in biology and mathematics. A review of such studies can be found in [2] and references therein. However it was understood for a long time that a classical approach would not be able to reproduce the phenomenon of like-charge attraction. It turns out that classically, the ions in the solution have at most a screening effect and cannot produce novel phenomena observed such as overscreening; the phenomenon of counter-ions over-compensating for the charge of a charged macro-ion[1]. Perturbative calculations have indeed shown that statistical fluctuations can contribute to an attractive interaction between like-charges[3]. However, the attractive contribution is a subleading effect in the perturbation theory and the overall force in the perturbative regime could not be attractive.

More recent studies have shown that overall attractive forces for like-charges can be reproduced when one accounts for the statistical fluctuations. For example in [4] the authors probe the interaction between two plates with equal charges in a 'strong coupling regime'–when the plate separation is small–and show that they attract. It has also been argued that the counter-ions in the solute form a Wigner Crystal-like structure on the surface of the DNA. In [5] and references therin, the case is made that when two DNA molecules come close, an attractive force arises due to the rearrangement of the structure of these crystals.

The surprising fact is that the microscopic field theory models that are used to describe statistical fluctuations in the literature strongly resemble a deformed version of Liouville theory. Liouville theory is a theory of two-dimensional quantum gravity that has been studied in high energy theory in many different contexts. For a general review and extensive references, consult [6]. The purpose of this paper is to describe how this is so and suggest interesting connections between two distinct fields of study–biophysics and quantum gravity.

To do so, we first review the field theory model used to describe the statistical physics of the electrostatics of solutes. Then we describe how it is related to Liouville theory. Next we make a motivated assumption that the theory can be approximated by a deformed version of Liouville theory and analyse the UV and IR behavior of like-charge DNA interactions. We conclude by summarizing the findings, pointing out some subtleties of the analysis, and speculating on future directions.

FIELD THEORY MODEL

We review the field theory used to describe charged solutes following [7]. There the authors begin with a many-body partition function of N large charged particles in a background of $i = 1, \cdots, n$ species of fluctuating ions. The charges of ions are given by z_i in unit charge. After a Hubbard-Stratonovich transformation they obtain the partition function,

$$Z = \int \frac{\mathcal{D}\phi}{Z_0} \exp\left[- \int d^3 r \left[\frac{(\nabla\phi)^2}{8\pi l_B} + \sum_i n_i \log\left(\int \frac{d^3 r'}{V} e^{-i z_i \phi(r')} \right) - \sum_i n_i \log\left(\frac{n_i}{e n_i^0} \right) - i \sum_j Q_j \delta^3(r - R_j) \phi(r) \right]\right] \tag{3}$$

in the presence of large objects at positions R_j with charge Q_j. $l_B = e^2/4\pi\epsilon k_B T$ is the Bjerrum length and charges are given in units of unit charge e. Z_0 is the determinant of the operator v. ϕ is a dimensionless field.

We focus on situations that are effectively two-dimensional. Happily, this is possible as DNA molecules can be well approximated by thin long rods[1]. We assume the charges are produced by parallel DNA molecules and only consider fluctuations that are translation invariant along the aligned directions. Therefore we assume that all sources are line-like and integrate along the aligned direction. Also we integrate out the density fluctuations n_i. Taking λ to be the length along the DNA that contains unit (negative) charge, i.e. the inverse line charge density, the final expression obtained is

$$Z = \int \frac{\mathcal{D}\phi}{Z_0} \exp\left[-\frac{1}{4\pi} \int d^2 r \left[(\nabla\phi)^2 - \sum_i (4\pi n_i^0 \lambda) e^{-i z_i \nu \phi} \right]\right] \prod_{j=1}^{N} e^{i\nu\phi(R_j)} \tag{4}$$

where we have defined the constant, $\nu = \sqrt{2 l_B/\lambda}$. It is worth noting that this action reproduces the Poisson-Boltzmann equation (1), (2) upon analytic continuation of ϕ, and that it was enough to demonstrate attraction between like-charged plates at strong-coupling[4].

CONNECTION TO LIOUVILLE THEORY

The action (4) strongly resembles correlators in Liouville theory. The action for Liouville theory is given by

$$S = \frac{1}{4\pi} \int d^2 z \sqrt{g} (g^{ab} \partial_a \phi \partial_b \phi + Q R \phi + 4\pi\mu e^{2b\phi}) \tag{5}$$

where R is the Ricci scalar in two dimensions. It is parameterized by two parameters μ and b. Q is given by $Q = b + 1/b$. In flat space this is reduced to

$$S = \frac{1}{4\pi} \int d^2 z (\partial_a \phi \partial^a \phi + 4\pi\mu e^{2b\phi}) \tag{6}$$

This is a conformal theory, and hence the correlators of the operators are restricted by the conformal symmetry. In particular for two operators $\mathcal{O}_1, \mathcal{O}_2$ with conformal dimension Δ_1, Δ_2,

$$\langle \mathcal{O}_1(z) \mathcal{O}_2(z') \rangle \propto \frac{\delta_{\Delta_1 \Delta_2}}{|z - z'|^{2\Delta_1}} \tag{7}$$

Notice that the position dependent part of the effective potential is given by,

$$U(z, z') = 2\Delta_1 \ln |z - z'| \tag{8}$$

which is a Coulumb potential. The operators $e^{2\alpha\phi}$ have conformal dimension $\Delta = 2\alpha(Q - \alpha)$. It is easy to see that $e^{2\alpha\phi}$ and $e^{2(Q-\alpha)\phi}$ have the same dimension and therefore,

$$\langle e^{2\alpha\phi(z)} e^{2(Q-\alpha)\phi(z')} \rangle \propto \frac{1}{|z - z'|^{2\alpha(Q-\alpha)}} \tag{9}$$

Now according to (4), a solute with only one type of counterion has an action that strongly resembles the Liouville action. The difference is that the operators active in the theory are imaginary exponents of ϕ. In fact string theorists do actually analytically continue the parameter b to an imaginary value and at the same time continue $\phi \to -i\varphi$ to obtain a 'time-like Liouville theory[8].' This precisely reproduces the Poisson-Boltzmann action. This theory has a kinetic term of the wrong sign but its correlators are defined via analytic continuation of the Liouville correlators.

The partition function (4) might not be a Liouville correlator even if there is only one kind of counterion in the solute, if the central charge of the theory $Q \neq (-iz\nu) + 1/(-iz\nu)$. But we may assume that it is a Liouville theory that is deformed by some operator $e^{-iz\nu\phi}$ and see how far we can go. Assuming that Z in (4) is a correlator of a deformed time-like Liouville theory, and assuming that it is sensible to define it as an analytic continuation of Liouville theory, one might ask what the central charge of the theory should be. Let us take the central charge of the initial theory we are deforming to be $Q = i\Lambda_0$ for some Λ_0. As we work in flat space it is not clear what the central charge is from the Lagrangian. The physics should dictate what Λ_0 should be.

We will assume that the theory has an IR and UV conformal fixed point with central charges $i\Lambda_{IR}$ and $i\Lambda_{UV}$. This is not such a drastic assumtion for a two-dimensional theory. We will be interested in a two-DNA interaction. It is clear that the effective action becomes

$$-\ln\langle e^{i\nu\phi(z)}e^{i\nu\phi(z')}\rangle = \nu(2\Lambda-\nu)\ln|z-z'| \tag{10}$$

up to a constant.

Now $e^{-iz\nu\phi}$ is an operator of dimension $\Delta = z\nu(\Lambda_0+z\nu/2)$. It is marginal when $\Delta=2$ and relevant(irrelevant) when it is smaller(larger) than 2. In the case this operator is marginal, the action (4) becomes exactly the analytically continued Liouville action. Also, the operator becomes dominant in the IR(UV) when $\Lambda_0<\Lambda'(\Lambda_0>\Lambda')$ where we have defined,

$$\Lambda' \equiv \frac{2}{z\nu}-\frac{z\nu}{2} \tag{11}$$

So we may make the most naive guess that,

- **Case 1:** When $\Lambda_0<\Lambda'$, $\Lambda_{IR}=\Lambda'$, $\Lambda_{UV}=\Lambda_0$
- **Case 2:** When $\Lambda_0>\Lambda'$, $\Lambda_{IR}=\Lambda_0$, $\Lambda_{UV}=\Lambda'$

Motivated by the observed phenomena, let us impose the following conditions. Let us demand that we have a screened repulsive potential in the IR:

$$0<\nu(2\Lambda_{IR}-\nu)<\nu^2 \tag{12}$$

The first inequality comes from demanding a repulsive potential, the second comes from assuming screening. This leads to

$$\frac{\nu}{2}<\Lambda_{IR}<\nu \tag{13}$$

Then let us distinguish between two UV behavior:

- **Case A:** An attractive potential in the UV, i.e. $\nu(2\Lambda_{UV}-\nu)<0$.
- **Case B:** A repusive potential in the UV, i.e. $\nu(2\Lambda_{UV}-\nu)>0$.

The allowed parameter space for the given four cases are,

- **Case 1A:** $\frac{2}{\sqrt{z(z+2)}}<\nu<\frac{2}{\sqrt{z(z+1)}}$, $\Lambda_0<\frac{\nu}{2}$.
- **Case 1B:** $\frac{2}{\sqrt{z(z+2)}}<\nu<\frac{2}{\sqrt{z(z+1)}}$, $\frac{\nu}{2}<\Lambda_0<\Lambda'$.
- **Case 2A:** $\frac{2}{\sqrt{z(z+1)}}<\nu$, $\frac{\nu}{2}<\Lambda_0<\nu$.
- **Case 2B:** $\frac{2}{\sqrt{z(z+2)}}<\nu<\frac{2}{\sqrt{z(z+1)}}$, $\Lambda'<\Lambda_0<\nu$.

In particular, it is clear that in order for there to be a repulsive potential in the UV(case B,) ν must satisfy

$$\frac{2}{\sqrt{z(z+2)}}<\nu<\frac{2}{\sqrt{z(z+1)}} \tag{14}$$

compared to the full range of allowed ν values, $\nu\geq 2/\sqrt{z(z+2)}$.

The range of ν which are admissable, and the range of ν that give repulsive interactions in the UV for each z are given in table I. It is clear that when $1.15<\nu<1.41$, the interaction in the UV can be repulsive only for $z=1$. So when ν is in this range, the UV interaction can be repulsive only when the counter-ions have charge 1. Solutions with different counter-ion charges will exhibit attractive interactions in the UV.

z value	Allowed	Repulsive
1	≥ 1.15	(1.15,1.41)
2	≥ 0.70	(0.70,0.81)
3	≥ 0.52	(0.52,0.58)
4	≥ 0.41	(0.41,0.45)

TABLE I. The range of ν which are admissable, and the range of ν that give repulsive interactions in the UV for each z.

For the value $\epsilon\approx 80$ for water and $T\approx 300K$ and $\lambda=0.17$nm, the value of ν is given by, $\nu\approx 3.0$. Actually, since the dielectric constant of charged solutions are much lower than that of water so ν is expected to be very large compared to the values given in table I. So according to this calculation all solutions should have attractive forces in the UV under real-world conditions. This situation could be alleviated by introducing many different counterions in the solution. With more parameters to tune, we expect to be able to engineer a model with realistic parameter ranges.

It is quite surprising that our naive assumptions led to a semi-realistic model of electrostatics of solutes. In particular, it is interesting that tuning ν makes the electrostatic force on the DNA's qualitatively differ for solutions with different counter-ion charges. However, we should acknowledge that we have made some jumps without carefully justifying them to reach this conclusion. We will end this section by listing the assumptions we have made. Hopefully a careful treatment of these points will lead to a clearer picture of why our model has some attractive qualitative features.

1. We have assumed that the statistical theory of the fluctuating electric potential is equivalent to a deformation of a time-like Liouville theory. Furthermore, we have assumed the correlators of the theory could be sensibly obtained by analytic continuation. This point should be more carefully justified.
2. We have assumed that the UV and IR fixed points can be described by a time-like Liouville theory. A clearer argument for this is desirable.

3. The claim that there is an attractive potential in the UV should be framed more carefully. From the Liouville theory point of view, what is actually happening is that the operators that were well defined in the IR/intermediate regime are not well defined operators in the UV anymore, i.e. there is an instability in the UV. From the field theory point of view, the IR description is breaking down and interactions must be described in terms of UV physics.

CONCLUDING REMARKS

There is a remarkable resemblance between the statistical model of the fluctuating electric potential and timelike Liouville theory. Inspired by this, we have boldly assumed that the statistical model of DNA's in a solute of counter-ions can indeed be approximated by a deformed timelike Liouville theory and examined the UV and IR behavior. Surprisingly this model exhibited some desirable qualitative features. In particular for a certain range of $\nu \equiv \sqrt{2l_B/\lambda}$ ($\nu \sim 1$) like-charged DNA molecules repelled each other with a screening in the IR, and attracted each other for counterion charge $z > 2$ and repelled each other for $z = 1$ in the UV. Although in real-world conditions ν is expected to be much larger, we expect to be able to reproduce the value of ν by taking into account more counterions.

However there are many strong assumptions made in this process that are far from being justified. We have summarized this at the end of the previous section. We expect to gain insight into why the model has nice qualitative features by examining the validity of each assumption.

Electrostatics of charged solutions and quantum Liouville theory are subjects that are still far from being well understood. It is exciting that two theories that have been intensely studied from different points of view seem to be deeply related to each other. It would be interesting to gain insight into both realms of physics through the looking glass of one another.

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