4.5 Synchronization

We have established how simple chemical reactions can create an oscillator. The phase of such simple oscillators is set by initial conditions. In certain situations, e.g. for the pacemaker cells in the heart, the oscillators must act in concert. One way to synchronize a number of oscillators is to couple them to an external source, such as the light of the sun in the example of circadian rhythms. In other cases, a collection of self–coupled oscillators may spontaneously synchronize as a collective. The following *Kuramoto model* provides an explanation of how such synchronization may happen.

Consider a set of oscillators, each parametrized by a phase angle θ_i, for $i = 1, 2, \cdots, N$. We shall assume that each oscillator advances at a uniform angular velocity ω_i, taken independently from a probability distribution function $p(\omega)$, such that

$$\dot{\theta}_i = \omega_i.$$

In a biological context the rate ω for a collection of cells (or organisms) may well depend on the concentration of chemicals within each. While these concentrations vary between individuals, it is likely that for a specific system the range of this variation is small, and the distribution is narrowly peaked around some central frequency Ω. Without loss of mathematical rigor we can set $\Omega = 0$, which is equivalent to measuring angles relative to a frame rotating at angular velocity Ω (i.e. after a shift $\theta_i \to \theta_i - \Omega t$).

To synchronize the oscillators we need a coupling between their phases. The simplest form of such coupling, pushing θ_i towards θ_j, and independent of a phase change by 2π, is $\sin(\theta_j - \theta_i)$. The coupled dynamics of phases of many such oscillators is now governed by

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} W_{ij} \sin(\theta_j - \theta_i)\,, \tag{4.39}$$

where W_{ij} indicates the strength of the coupling to j from i. To make analytical progress, we shall assume that all interactions have the same value of K/N. (As each oscillator is coupled to N others, it makes sense to scale the interaction parameter by $1/N$, corresponding to an average.) In this case, we can re-write Eq. (4.39) as

$$\begin{aligned} \dot{\theta}_i &= \omega_i + K\Im\left[e^{-i\theta_i}\left(\frac{\sum_{j=1}^{N} e^{i\theta_j}}{N}\right)\right] \\ &= \omega_i + K\Im\left[e^{-i\theta_i}\cdot me^{i\phi}\right]\,, \end{aligned} \tag{4.40}$$

where $\Im$ stands for the imaginary part, and we have indicated the average of all phase points (around the complex imaginary circle) by $me^{i\phi}$. The *order parameter*

$$m = \left|\frac{\sum_{j=1}^{N} e^{i\theta_j}}{N}\right|\,, \tag{4.41}$$

is a measure of synchronization amongst the oscillators. If each oscillator follows its own period, perhaps somewhat altered by the others, the phases in Eq. (4.41) will shift with time more or less independently, eventually covering the unit circle, in which case the average over them will be zero. If a finite fraction of the oscillators is locked to the central frequency Ω (thus appearing stationary in our rotating frame), their contributions will be time independent, and (if more or less in phase) add up to a finite value. Without loss of generality, we can set the overall phase of the sum to zero, $\phi = 0$, resulting in the self-consistent set of equations

$$\dot{\theta}_i = \omega_i - Km \sin\theta_i \,. \tag{4.42}$$

The solutions to this equation have two possible forms, mimicking the two populations of oscillators. The first set includes oscillators locked to the central frequency, hence with $\dot{\theta}_i = 0$. To satisfy Eq. (4.42), these oscillators acquire a "phase lag"

$$\sin\theta_i = \frac{\omega_i}{Km} \,; \tag{4.43}$$

oscillators faster than Ω are ahead of the pack, those with $\omega < \Omega$ fall behind. Such locking is possible only if the natural frequency of the oscillator is sufficiently close to the central frequency, i.e. as long as $|\omega_i| < Km$. Oscillators with frequency difference $|\omega_i| > Km$ cannot be synchronized to the central frequency, and their phases vary over time according to

$$\dot{\theta}_i = \omega_i - Km \sin(\theta_i) \neq 0 \,. \tag{4.44}$$

We can self-consistently solve for m by summing over the phase contributions of the stationary oscillators (the moving ones do not contribute to m). Since the behavior of the locked oscillators depends only on their native frequency, we can use the probability density $p(\omega)$ to write

$$m = \frac{1}{N} \sum_{\text{locked oscillators } j} e^{i\theta_j} = \int_{-Km}^{Km} d\omega \, p(\omega) e^{i\theta(\omega)} \,. \tag{4.45}$$

We can change variables from $\omega = Km \sin\theta$ to $\theta = \arcsin(\omega/Km)$, and expand the narrow distribution to second order around its peak to simplify the self-consistency equation to

$$\begin{aligned} m &= \int_{-\pi/2}^{\pi/2} d\theta (Km\cos\theta) p(Km\sin\theta) e^{i\theta} \\ &= \int_{-\pi/2}^{\pi/2} d\theta (Km\cos\theta) \left[p(0) - \frac{(Km\sin\theta)^2}{2} |p''(0)| + \cdots \right] (\cos\theta + i\sin\theta) \\ &= Km \left[\frac{\pi}{2} p(0) - \frac{K^2 m^2 \pi}{8} |p''(0)| + \cdots \right] \,. \end{aligned} \tag{4.46}$$

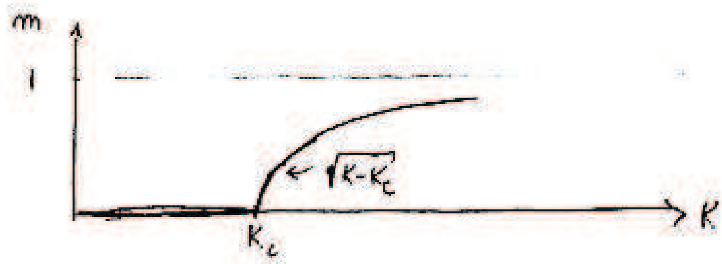
For example, for a Gaussian distribution of variance σ^2, $p(0) = 1/\sqrt{2\pi\sigma^2}$ and $|p''(0)| = p(0)/\sigma^2$.

A non-zero solution for m is possible only for

$$K > K_c = \frac{2}{\pi p(0)} \,. \tag{4.47}$$

Below K_c, $m = 0$ is the only solution, and all oscillators are unlocked. For larger couplings the oscillators are synchronized and rotate together, while on reducing the coupling to its critical value the order parameter vanishes as

$$m \propto \sqrt{K - K_c}\,. \tag{4.48}$$



For example, if $p(\omega)$ is approximated by a Gaussian distribution of width σ,

$$K_c = \sqrt{\frac{8}{\pi}}\sigma \,, \quad \text{and} \quad m \simeq \sqrt{2\pi\left(\frac{K}{K_c} - 1\right)} \,. \tag{4.49}$$

As an alternative example, you may work through the case where the frequencies are uniformly distributed in the interval $[\Omega - \omega_m, \Omega + \omega_m]$.