MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.962: General Relativity — February 16, 2018

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PROBLEM SET 2

DUE DATE: Thursday, February 22, 2018, at 5:00 pm.

TOPICS COVERED AND RELEVANT LECTURES: This problem set covers metrics, connections, and geodesics for Riemannian spaces, primarily following material that was or will be presented in Lectures 3 (Wed 2/14) and 4 (Tues 2/20). This material diverges from the order of presentation in the texts of Carroll and Wald, where connections and geodesics are introduced only after a more extended discussion of geometry. We will go further into the geometry underlying these structures in later weeks.

MAXIMUM GRADE: This problem set has a total of 50 points.

PROBLEM 1: GEODESICS ON THE SURFACE OF A SPHERE *(10 pts)*

In this problem we will test the geodesic equation by computing the geodesic curves on the surface of a sphere. The metric on the surface can be written in terms of the usual polar angles θ and ϕ, where θ is the angle from the z-axis, and ϕ is the azimuthal angle. If the radius of the sphere is a, the metric is given by

$$ds^2 = a^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) \quad .$$

(a) Clearly one geodesic on the sphere is the equator, which can be parametrized by $\theta = \pi/2$ and $\phi = \psi$, where ψ is a parameter that runs from 0 to 2π. Show that if the equator is rotated by an angle α about the x-axis, then the equations become:

$$\begin{aligned} \cos\theta &= \sin\psi \sin\alpha \\ \tan\phi &= \tan\psi \cos\alpha \quad . \end{aligned}$$

(b) Using the geodesic equation, derive the differential equation which describes geodesics in this space. I recommend using the "primitive" form of the geodesic equation,

$$\frac{d}{d\psi}\left[g_{k\ell} \frac{dx^\ell}{d\psi} \right] = \frac{1}{2} \frac{\partial g_{ij}}{\partial x^k} \frac{dx^i}{d\psi} \frac{dx^j}{d\psi} \ ,$$

but feel free to use the "standard" form,

$$\frac{d^2 x^\ell}{d\psi^2} = -\Gamma^\ell_{ij} \frac{dx^i}{d\psi} \frac{dx^j}{d\psi}$$

if you prefer. Both of these formulas are valid only if ψ is proportional to the length along the geodesic. Is it?

(c) Show that the expressions in (a) satisfy the differential equation for the geodesic. Hint: The algebra on this can be messy, but I found things were reasonably simple if I wrote the derivatives in the following way:

$$\frac{d\theta}{d\psi} = -\frac{\cos\psi\sin\alpha}{\sqrt{1-\sin^2\psi\sin^2\alpha}} \ , \qquad \frac{d\phi}{d\psi} = \frac{\cos\alpha}{1-\sin^2\psi\sin^2\alpha} \ .$$

PROBLEM 2: GEODESICS IN POLAR COORDINATES *(10 pts)*

Considered the metric $ds^2 = dr^2 + r^2 d\theta^2$, which describes 2D flat (i.e., Euclidean) space in polar coordinates. By relating these coordinates to the Cartesian coordinates, determine the geodesic trajectory $(r(\lambda), \theta(\lambda))$ from the point $(r, \theta) = (1, 0)$ to the point $(2, \pi/3)$. Confirm that this trajectory satisfies the geodesic equation

$$\frac{d^2 x^\ell}{d\lambda^2} = -\Gamma^\ell_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda} \, .$$

(In this case you should use the "standard" geodesic equation, as written above, to make sure that you get some experience in calculating Christoffel symbols.)

PROBLEM 3: A METRIC IN THE PLANE, INTUITION FOR THE SHORTEST PATH *(15 pts)*

Consider the metric on the plane $\mathbb{R}^2$ given by

$$ds^2 = (1 + 3x^2)^2(dx^2 + dy^2)$$

Note that this metric is like the usual Euclidean metric, but scaled by an overall factor that depends on position.

(a) Compute the distance between the points $p = (-1, -2)$ and $q = (1, 2)$ computed along a linear trajectory connecting the two points.

(b) Compute the geodesic equation, in either of the forms that we have discussed, and show that the straight line trajectory (even with proper parameterization) does not satisfy the geodesic equation.

(c) Find a trajectory composed of 3 linear pieces along which the distance is shorter than the straight line path.

[Hint: think about where in the plane the metric scaling factor is smallest, making it "cheapest" to have the path go in terms of overall length.]

(d) Sketch what you think the geodesic connecting these two points would look like (you do not need to solve the geodesic equation exactly.)

PROBLEM 4: THE HYPERBOLIC PLANE *(15 pts)*

Consider the metric on the upper half plane $\{(x, y) : y > 0\}$ given by

$$ds^2 = \frac{1}{y^2}(dx^2 + dy^2).$$

This is a classic example of a curved space, known as the *hyperbolic plane*, that we will return to later.

(a) Compute the length of a path parallel to the y-axis that reaches from the point (x, y) to the point $(x, 0)$. Can you see why the x-axis is not considered part of this geometry?

(b) Compute the components of the Christoffel connection and write the geodesic equation. You may write the geodesic equation in either the "standard" or "primitive" forms (as defined in Problem 1), but calculate the Christoffel symbols in any case, just for practice.

(c) Show that a path parallel to the y-axis (with proper parameterization) solves the geodesic equation.

(d) Show that a semicircle centered at a point $(x, 0)$ on the x-axis is a geodesic trajectory.

[Note: this part may require a bit of work to show, but can be done in a variety of ways. If you are stuck, brainstorm with your classmates. If you are still stuck, we can give a hint.]

This example played an important role in the development of non-Euclidean geometry. Note that Euclid's fifth (parallel) postulate is violated: given a straight line L (*e.g.* the upper half of the y-axis), and a point p (*e.g.* $(1, 1)$), there are *an infinite number* of geodesics that pass through p but do not intersect L.