MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department

Physics 8.962: General Relativity — April 19, 2018
Prof. Alan Guth

PRELIMINARY PROBLEM SET 10

DUE DATE: Thursday, April 26, 2018, at 5:00 pm.

TOPICS COVERED AND RELEVANT LECTURES: This problem set covers the Tolman-Oppenheimer-Volkoff equation, as discussed in Carroll's Chapter 5, and as will be discussed in lecture on Monday April 23. Part (a) of the problem requires some basic quantum theory, to calculate the energy density and pressure of a degenerate gas of non-interacting neutrons. Nikhil Raghuram will discuss this subject in recitations on Friday and Monday. In addition, I will post a section from Griffith's **Introduction to Quantum Mechanics** about this subject.

PRELIMINARY?!: The full problem set, to be posted shortly, will probably include one more problem, most likely about the use of Kruskal coordinates.

PROBLEM 1: SOLVING THE TOV EQUATION FOR A DEGENERATE NEUTRON GAS *(30 pts)*

Consider a neutron star composed of a degenerate gas of non-interacting neutrons, which obey Fermi statistics. We will assume a central density which is a constant X times nuclear density, which we take as

$$\rho_N = \frac{1}{3}\frac{8\pi m_N^4}{(2\pi\hbar)^3} \,. \tag{1}$$

(a) [10 pts] Express the energy density and pressure in terms of the Fermi momentum k_F. This gives an equation of state $p(\rho)$ for the neutron matter, since ρ determines k_F, which in turn determines p. Remember that the neutron has two spin states, and use relativistic mechanics. (This of course is quantum mechanics, and not general relativity, but it is probably something that you know how to do. If not, review your quantum mechanics or ask at office hours.)

(b) [8 pts] For convenience, define dimensionless variables

$$\tilde\rho \equiv \frac{\rho}{\rho_N} \,, \quad \tilde p = \frac{p}{\rho_N} \,, \tag{2}$$

$$\tilde k = k_F/m_N \,, \quad \tilde r = r\sqrt{G\rho_N} \,, \quad \tilde m(\tilde r) = m(\tilde r) G\sqrt{G\rho_N} \,. \tag{3}$$

Use the Tolman-Oppenheimer-Volkoff equation to derive an equation for

$$\frac{\mathrm{d}\tilde k}{\mathrm{d}\tilde r} \,. \tag{4}$$

Recall that the TOV equation is given by

$$\frac{dp}{dr} = -\frac{(\rho + p)[Gm(r) + 4\pi G r^3 p]}{r[r - 2Gm(r)]} \ . \tag{5}$$

where

$$m(r) = 4\pi \int_0^r \rho(r')\, r'^2 \, \mathrm{d}r' \ . \tag{6}$$

(c) [12 pts] For the special cases of $X = 0.5, 1, 2$, and 4, integrate the TOV equations to find the mass and radius of the star. Express the mass in terms of solar masses, and express the radius in kilometers. (Hint: we recommend using $\tilde{k}(\tilde{r})$ as the function to integrate, since it is easier to express $\tilde{P}$ as a function of $\tilde{k}$ than vice versa. One complication is that the integration does not extend to some fixed maximum value of $\tilde{r}$, but rather should stop when $\tilde{k}$ reaches zero. If you use Mathematica, you should be aware that the "Method" called "EventLocator" is designed to handle exactly this sort of situation.)