5.1.2 Steady state

Because of the conservation of probability in Eqs. (5.1.2) and (5.1.13), the transition probability matrix π, and by extension the rate matrix $\mathbf{R}$ have a left-eingenvector $\overleftarrow{v^*} = (1, 1, \cdots, 1)$ with eigenvalues of unity and zero respectively, i.e.

$$\overleftarrow{v^*}\pi = \overleftarrow{v^*} \quad , \quad \text{and} \quad \overleftarrow{v^*}\mathbf{R} = 0. \tag{5.1.4}$$

For each eigenvalue of a matrix there is both a left eigenvector and a right eigenvector. The matrices π and $\mathbf{R}$ thus must also have a right-eigenvector $\overrightarrow{p^*}$ such that

$$\pi\overrightarrow{p^*} = \overrightarrow{p^*} \quad , \quad \text{and} \quad \mathbf{R}\overrightarrow{p^*} = 0. \tag{5.1.5}$$

The elements of the vector $\overrightarrow{p^*}$ represent the *steady state probabilities* for the process. These probabilities no longer change with time. In many cases, the Perron–Frobenius theorem ensures that the vector $\overrightarrow{p^*}$ is unique, with positive elements as appropriate to probabilities.[1] Since the matrix π is not symmetric, the remaining eigenvalues need not be real, but must occur in complex conjugate pairs. These remaining eigenvalues of the transition matrix have magnitude less than unity, and determine how an initial vector of probabilities approaches the steady state.

From Eq. (5.1.3) we observe that a particular steady state $\overrightarrow{p^*}$ can be achieved through transition probabilities that satisfy the so-called condition of *detailed balance*,

$$\pi_{\alpha\beta}p^*_\beta = \pi_{\beta\alpha}p^*_\alpha\,. \tag{5.1.6}$$

This condition holds for systems in thermal equilibrium, where the steady-state probabilities are constrained to satisfy the Boltzmann distribution ($p^*_\alpha \propto \exp(-\beta E_\alpha)$). Equation (5.1.6) requires equal probability fluxes between any pair of states. More general steady states can be formed with the probability flux circulating along triplets and larger sets. Indeed, such circulation of flux can be used as indicator of non-equilibrium steady states.

[1]A condition for the validity of the above is that it any state should be accessible (through a set of transitions) from any other states. Consequences of violation of this condition will be explored later in connection with genetic drift and absorbing states.