1. **Maximally entangled states:** In this problem, we will work with a generalization of the EPR state called the *maximally entangled state*. Consider the state space $\mathbb{C}^d \otimes \mathbb{C}^d$, and denote the standard basis of $\mathbb{C}^d$ by $\{|1\rangle, \ldots, |d\rangle\}$. The *maximally entangled state* in this space is defined to be

$$|\Phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle \otimes |i\rangle.$$ 

(a) Show that for any $d \times d$ matrix $A$, it holds that

$$A \otimes I |\Phi\rangle = I \otimes A^T |\Phi\rangle,$$

where $A^T$ is the transpose of $A$. (Extra food for thought: is the transpose basis-dependent?)

(b) Show that for any two $d \times d$ matrices $A$ and $B$, it holds that

$$\langle \Phi | A \otimes B |\Phi\rangle = \frac{1}{d} \text{tr}(AB^T).$$

(c) Show that for any orthonormal basis $\{|v_1\rangle, \ldots, |v_d\rangle\}$ of $\mathbb{C}^d$, the maximally entangled state can be expressed as

$$|\Phi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |v_i\rangle \otimes |v_i^*\rangle,$$

where $|v_i^*\rangle$ is the complex conjugate of the vector $|v_i\rangle$.

2. **Stabilizers:** Recall the Pauli $X$ and $Z$ matrices from class

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ 

(a) Write an eigendecomposition for $X \otimes X$ and $Z \otimes Z$.

(b) A state $|\psi\rangle$ is *stabilized* by an operator $M$ if $M |\psi\rangle = |\psi\rangle$. Write down the states stabilized by

i. $X \otimes I$ and $I \otimes Z$.
ii. $X \otimes X$ and $Z \otimes Z$.
iii. $X \otimes X$ and $-Z \otimes Z$.

(c) Is there a state stabilized by $X \otimes X$ and $Z \otimes I$? If not, why not?
(d) (Optional:) Suppose that $\langle \psi | (X \otimes X + Z \otimes Z) | \psi \rangle \geq 2 - \epsilon$. Find a bound on the minimal Euclidean distance $\min_\theta \| e^{i\theta} | \psi \rangle - |EPR\rangle \|$ between a state that is a multiple of $|\psi\rangle$ and the EPR state, as a function of $\epsilon$. (Hint: consider the eigendecomposition of the matrix $X \otimes X + Z \otimes Z$.)

3. **The GHZ game:** In this problem, we will introduce tripartite states, corresponding to three quantum systems. Suppose Alice, Bob, and Charlie each have a single qubit. Then their joint state space is $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$. As usual, we denote the standard basis of $\mathbb{C}^2$ by $\{|0\rangle, |1\rangle\}$. $X$ and $Z$ are the Pauli matrices as in the previous problem.

(a) The **GHZ state** is the following entangled state

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle \otimes |1\rangle).$$

(b) Write down all tensor products of $X$, $Z$, and the identity $I$ that stabilize $|GHZ\rangle$. You should find five such matrices, including $I \otimes I \otimes I$.

(c) Suppose Alice and Bob have lost contact with Charlie. Show that nevertheless they can distinguish between the GHZ state and the following state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)_{AB} \otimes |1\rangle_C.$$

Do this by finding an observable $O$ acting on Alice and Bob’s systems such that

$$\langle \psi | O \otimes I | \psi \rangle \neq \langle GHZ | O \otimes I | GHZ \rangle.$$

(Hint: consider a tensor product of $X$ or $Z$ matrices).

(d) In the **GHZ game**, Alice, Bob, and Charlie are separated so that they cannot communicate, and play together against a referee. The referee samples a triple of bits $(x, y, z)$ from $\{(0,0,0), (0,1,1), (1,0,1), (1,1,0)\}$ uniformly at random, and sends $x$ to Alice, $y$ to Bob, and $z$ to Charlie. Each player responds with a single-bit answer; we denote Alice, Bob, and Charlie’s answers by $a, b,$ and $c$ respectively. The players win if $x \lor y \lor z = a \oplus b \oplus c$.

i. What is the maximum probability of winning for Alice, Bob, and Charlie if they use a classical strategy?

ii. Describe a quantum strategy for the players to win the game with certainty. (Hint: use the GHZ state, and the stabilizers you found in the first part of the problem.)