APPLIED SYSTEMS ANALYSIS

Engineering Planning and Technology Management

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CHAPTER 20

MULTIATTRIBUTE UTILITY

20.1 CONCEPT

This chapter presents Keeney-Raiffa multiattribute utility $U(\mathbf{X})$ as the recommended analytic method for measuring preferences of consequences with many different dimensions, $\mathbf{X} = (X_1, \ldots, X_i, \ldots, X_N)$. This approach is a direct extension of the one-dimensional utility presented in Chapter 19.

Keeney-Raiffa multiattribute utility is appropriate for practical analyses because it provides a good balance between convenience and ease of use on the one hand, and accuracy on the other. Compared to the elementary models, it is relatively fast while incorporating the kinds of interactive effects between dimensions that exist in interesting situations. Section 20.2 discusses the deficiencies of the alternative models in detail, setting the context for the subsequent presentation of the theory on which the preferred approach is based.

The essence of the theory lies in two assumptions, similar to axioms, about the nature of the preferences of a person or organization. These have a significant consequence: they reduce the number of calculations exponentially from the number required by a direct application of the methods for measuring utility presented in Chapter 19. Section 20.3 presents these assumptions and discusses their limits in detail. The following sections indicate how multiattribute utility should be used in practice, and illustrate its application.

Multiattribute utility complements the procedures presented in Chapter 8 for multiobjective analysis. Those methods, which seek to identify noninferior solutions, have the great advantage of not requiring any explicit measure of preference. They are therefore much simpler and should be used whenever possible.

The multiattribute methods should be used when there are no clearly dominant choices and when a more detailed, analytic method is required. In practical situations it is often most efficient first to use the multiobjective methods to screen out dominated, clearly unattractive possibilities and identify the noninferior alternatives as defined in Section 8.2. Then, if necessary because there is no dominant choice, the next step is to use multiattribute utility on the set of noninferior alternatives.

20.2 ELEMENTARY METHODS

There are two obvious ways of defining the preference function for a set of distinct objects or attributes. One is to add up the preference functions for the individual objects; the other is to extend the measurement of utility as described in Chapter 19. The first method often simplifies the situation excessively, thus neglecting many of the interesting features of the problem. The other requires far too much effort for practical use. Additionally, three other methods for evaluating multiattribute alternatives are popular because they are easy to use on personal computers. Yet these contain serious deficiencies. This section covers each of these approaches in detail, as a motivation for the recommended procedure.

Additive approach. This is the traditional method. It simply expresses:

$$U(\mathbf{X}) = \sum w_i U(X_i)$$

where the w_i are some scaling factors or weights between the values of the different dimensions.

This is the approach implicit in most practical economic analyses such as engineering economy (Chapter 12). In those cases the weights between the different objects are simply their prices, p_i. Since these approaches also assume that value is linear in quality, they express:

$$U(\mathbf{X}) = \sum p_i X_i$$

This simple formula is useful in the limited situations for which the assumptions hold, that is, when one is dealing in a strictly commercial environment, in which all X_i have prices; when these prices are constant regardless of the level of any X_i; and when these prices do in fact reflect the real value of the X_i. These situations occur in many routine operations. They are, however, quite atypical of the most interesting situations, particularly those involving large-scale systems.

The more sophisticated approach, of defining $U(\mathbf{X})$ as the weighted sum of the utilities for individual attributes $U(X_i)$, has a fundamental limitation that makes it unattractive in practice: the additive model cannot express the value of any interaction between the different objects. This is a major weakness because, as it happens, there is generally considerable value (either positive or negative) to the interaction between the several attributes of a system.

People value the interaction between the attributes of a system because these effects are frequently complementary: one or more attributes may have little value without others being present. Conversely, the attributes may also be antagonistic in that the presence of one attribute negates the value of the other. The whole may be either greater or less than the sum of its parts. Formally, any attribute X_i may have one value $U(X_i)'$ if another attribute X_j is present, and quite another, $U(X_i)''$ if X_j is absent (see box). It is in the nature of systems that the values of the attributes interact.

Additive models of utility cannot represent the broad range of common situations, and are therefore inadequate for general use. As the analysis for the two-dimensional case in Section 20.3 indicates, additive multiattribute utility is truly exceptional.

Extension of utility measurement. In principle, the utility over many attributes incorporating all relevant interactions, $U(\mathbf{X})$ can be measured just as the utility for any simple characteristic, $U(X_i)$. One could define the utility of two arbitrary reference points and then estimate the utility of all others with respect to them. This last step would require some special effort because one would be changing

Values for Complementary and Antagonistic Attributes

As an example of complementary attributes, consider a computer program that prepares graphic displays. It has two attributes that would obviously be interesting to any user:

$$X_1 = \text{Speed} \qquad X_2 = \text{Resolution}$$

A utility function can be defined for each of these attributes. These functions, $U(X_1)$ and $U(X_2)$, may be quite typical: a person will place considerable emphasis on the initial improvements in either speed or resolution, and will have diminishing marginal utility for ever better performance.

The additive model of value for both speed and performance is not appropriate, however:

$$U(\text{Speed, Resolution}) \neq w_1 U(\text{Speed}) + w_2 U(\text{Resolution})$$

Why? Because any graphics program must have both speed and resolution together; a program with great speed and no resolution is probably of little use, for example.

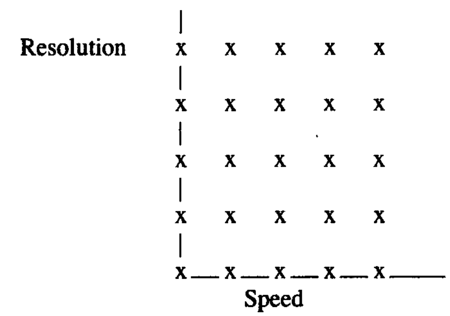
Examples of antagonistic attributes abound, although they may be less obvious. For example, you may value both spending time with your parents and going to parties. However, having your parents with you at a party with your peers may dampen your enjoyment considerably. Likewise, a community may value the acres available for bird sanctuaries and the land for airport development. Having the two together is rather worthless—the aircraft disturb the birds and the birds have the unfortunate tendency to get sucked into jet engines, causing crashes.

several attributes simultaneously and one would want to be careful to do so systematically. Yet there is no theoretical reason this could not be done.

The difficulty with this approach is practical. It requires such a large number of measurements that it is hardly feasible even for just two attributes, and certainly not for more. Indeed, the number of points that would have to be assessed by the direct application of utility measurement increases exponentially for any given degree of resolution with the number of attributes to be valued. Thus if one wants each single-dimensional utility function, $U(X_i)i = 1$ to N, to be defined by M points, the N-dimensional utility function $U(\mathbf{X})$ will have to be defined by about M^N points. More precisely, since we always assume the utility for two reference points, the number of points to be assessed by the direct extension of utility measurement is $(M^N - 2)$. (See box.)

Dimensionality of Direct Measurement of Multiattribute Utility

Suppose we wish to measure the utility of a graphics computer program considering the two attributes of speed and resolution. Suppose further that we will be satisfied with a crude measurement dividing the range of each dimension into quarters. The total number of points that must be defined is thus 25, as the display indicates:



Points, x, to be defined for two-dimensional utility.

As the value of two points could be assumed, such as $\mathbf{X}^*$, the best performance, and $\mathbf{X}_*$, the worst, the utility of 23 points ($= 5^2 - 2$) would have to be measured.

Both greater resolution and more dimensions would be impractical. Six points per dimension would require 34 points in all, at the limit of practicality; and a three-dimensional utility function would require $(5^3 - 2) = 123$ points, well beyond what is possible.

The practical limit to the number of points that can be measured in a utility function would appear to be about 30 if the assessment is conducted verbally. This limit may be as great as 50 if the assessments are done with an interactive computer program (see Section 19.3.) Beyond these limits the person being interviewed is too tired to respond meaningfully. These limits mean that only a two-dimensional multiattribute utility function can be measured by this method. The direct extension of utility measurement is therefore not a practical method for measuring multiattribute utility.

Lexicographic ordering. This approach to ranking consequences with many attributes derives its name from "lexicon," the name for a book that arranges things alphabetically, such as a dictionary or a telephone directory. This method operates on much the same principle.

The procedure involves two phases. The first is to rank the attributes in order of importance. The second is to rank the consequences one is evaluating in terms of their most important attribute, using the second most important to break ties between the rating on the first attribute, and so on (see box). This process is trivial to automate and thus commonly available on computers.

This mindless procedure is worthless for analysis. It ignores all possible trade-offs in values between the attributes. It implies that any advantage on the first attribute, however small, outweighs all advantages on the other attributes, no matter how large. It thus implies that all secondary attributes are worthless. If this is so, one might as well simply worry about one-dimensional utility. If we really do care about the many attributes of a consequence, this approach is totally unrealistic. The only reason to discuss it is to put you on guard against something that, looking easy, is seductive.

Lexicographic Ordering

Consider a series of graphics programs defined by their speed, in megahertz, and resolution, in terms of numbers of pixels to a side of the screen. Suppose that the (speed, resolution) characteristics of four of them are:

$$(16, 1000) \quad (9, 500) \quad (9, 250) \quad (6, 1000)$$

If speed is identified as the more important attribute, the lexicographic ordering would rank the programs in the order just given: the fastest first; the next fastest (9 MHz) according to their secondary attribute of resolution; and the slowest last.

Note that this procedure does not admit the possibility that the slow, high resolution program (6, 1000) could be better than the thoroughly mediocre program (9, 250). No trade-offs are considered and secondary attributes are virtually disregarded.

Conjunctive and disjunctive methods. These two methods are quite similar. Again, they proceed through two phases. The first is to define minimum criteria for each attribute. The second is to use these standards to define the set of things worth considering. Note carefully that, by itself, neither approach ranks the items in the set or defines the preferred choice—unless the set has only one element.

The *disjunctive* approach defines the set as all the elements that meet at least one criterion. It is the union of all the possibilities that have not been systematically rejected by each minimum standard. In terms of Figure 20.1, this is the set of elements that lie in the areas *A*, *B*, and *C*, between the minimum standards and the maximum possibilities as defined by the noninferior alternatives (see Section 8.2).

The *conjunctive* approach defines the set more narrowly as the conjunction of the possibilities that meet the criteria, that is, those in area *B*. This would seem the better method in that it could seem to define the preferred alternative more tightly.

Both approaches are easy to automate because they simply require a mechanical sort of alternatives according to their attributes, as with the lexicographic approach. Both are thus also readily available on computers and seductively easy to use. Both, however, fail equally to consider trade-offs between attributes and are of little value in practice.

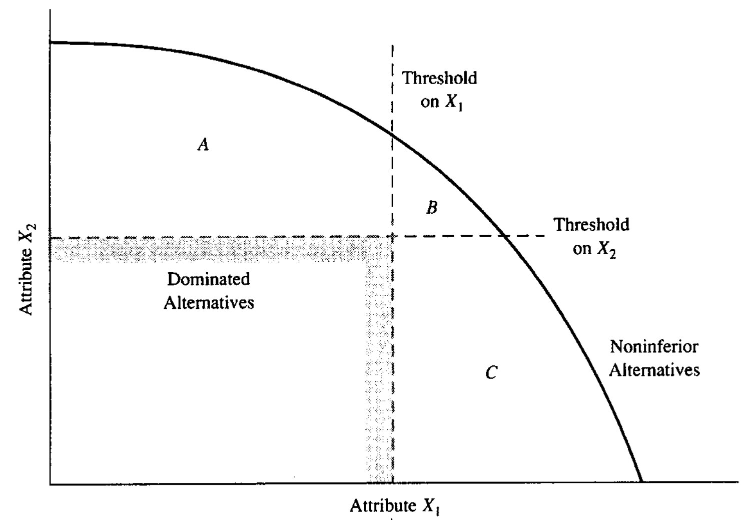


FIGURE 20.1
The disjunctive approach to multiattribute analysis includes alternatives in A, B, and C; the conjunctive approach focuses narrowly on alternatives in B.

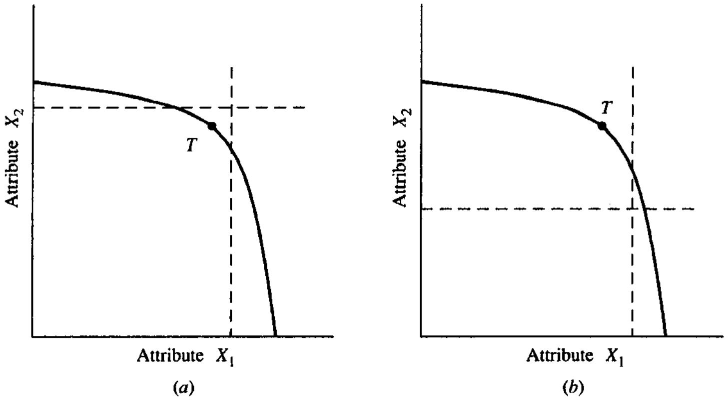


FIGURE 20.2
Situations in which the disjunctive (*a*) and conjunctive (*b*) approaches to multiattribute analysis miss the alternative *T*, which represents the best tradeoff between the different attributes.

The difficulties with these methods is easily seen by example. Consider the relatively common situation in which the curve of noninferior alternatives has a "knee" that defines the preferred choice at *T* (see discussion in Section 8.6). The first difficulty is that both the conjunctive and disjunctive methods may exclude the best choice, as Figure 20.2(*a*) shows. One might imagine that this problem could simply be resolved by lowering the standards and enlarging the sets. Such is not the case. Figure 20.2(*b*) provides such a situation: here the conjunctive approach does define a nonzero set of choices, but excludes the preferred one at *T*; the disjunctive includes *T*, but along with so many others as to not be especially useful.

In short, both the conjunctive and disjunctive methods are rather ineffective ways to define the set of alternatives one should focus on. This set consists of the noninferior alternatives as defined in Section 8.2 and is best defined by analysis or, if necessary, by sorting directly for noninferior alternatives.

20.3 THEORY

The theory of multiattribute utility is based on two assumptions about the structure of preferences. These assumptions refer to the way a person values the interaction between attributes. They lead to a considerable reduction in the number of calculations that are necessary. If the assumptions are acceptable for a person faced with a particular situation, we thus have a practical means to obtain the multiattribute utility while still accounting for the interaction between attributes.

The two assumptions are similar to axioms in that they define the nature of the analysis. They differ from axioms in the following way. Axioms are taken to be statements about how people should behave rationally. The special assumptions of multiattribute utility on the other hand do not have any ethical content, they are merely descriptive. The question to be asked concerning the assumptions is thus: Do these two assumptions describe the preferences of a person for a situation within a tolerable degree? If the answer is yes, then the multiattribute utility can be measured by the method to be described.

The two assumptions are those of

- preferential independence
- utility independence

The reason they simplify the measurement process is that they lead to ways to decompose the *N*-dimensional utility functions into *N* one-dimensional portions that can subsequently be recombined. The number of calculations is therefore changed from an exponential to a linear function of the number of attributes. Each assumption is described as follows.

Preferential independence. This assumption concerns the order of preferences. Specifically it is that the ranking of preferences over any pair of attributes is independent of the other attributes. Preferential independence is important because it permits us to decompose a multidimensional problem.

Formally, the assumption of preferential independence is that

- If for any pair of attributes, say X_1 and X_2, one combination is preferred to another

$$(X_1', X_2') \succ (X_1'', X_2'')$$

  for some level of the other attributes, say X_3 to X_N,
- then the order of this preference will be maintained for all other levels of the other attributes.

Symbolically, this can be written as (where *a*/*b* indicates conditionality and is to be read as "*a* exists when conditions *b* prevail")

If	$[(X_1^a, X_2^a)/(X_3, \ldots, X_N)] \succ [(X_1^b, X_2^b)/(X_3, \ldots, X_N)]$
implies	$[(X_1^a, X_2^a)/(X_3', \ldots, X_N')] \succ [(X_1^b, X_2^b)/(X_3', \ldots, X_N')]$,
then	X_1 and X_2 are preferentially independent of $(X_3, \ldots, X_N)$.

People are quite commonly preferentially independent between attributes (see box). Note however that being preferentially independent does not mean that one does not care about the other attributes. One probably does value them significantly. Preferential independence only implies that the order of ranking

Example of Preferential Independence

Again consider the computer graphics program. Most people would be preferentially independent between speed and resolution of the program on the one hand, and its cost on the other.

Thus a person interested in digital maps might prefer (medium speed, high resolution) to (high speed, low resolution) regardless of cost; the resolution would be particularly important. Thus,

If [(medium,high)/high cost] > [(high,low)/high cost]

implies [(medium,high)/low cost] > [(high,low)/low cost]

then the person is preferentially independent for speed and resolution versus cost.

The person presumably values cost as an important attribute of the choice, and would distinctly prefer a lower cost program. Preferential independence in this case implies that for any given price, the order in which pairs of levels of speed and resolution are ranked will stay the same.

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between two attributes does not change because of changes in the level—and value—of the other attributes.

Situations in which preferential independence does not hold are not common, but common enough that decision analysts should watch out for them carefully. Most people for example are not preferentially independent over the various attributes of their diet (see following box). More generally, there are many instances in which the ranking of one's preferences do depend on the level of some other attribute, particularly on whether it is above or below some threshold.

Analysts should investigate whether a person's preferences are in fact reasonably described by preferential independence between attributes. This must be done case by case, for each different person and for each specific kind of situation. The standard way to check for preferential independence is to ask a person at the start of the interview whether their ranking between attributes (such as speed and resolution for the computer graphics program) might depend on the other attributes. This should be done for all possible pairs of attributes (e.g., speed and resolution versus cost; speed and cost versus resolution; and so on.). This process is fairly tedious, but should be done as there may well be combinations for which preferential independence does not hold. For the computer graphics program it may not, for instance, if there is a minimum level of resolution that is required.

Situations in which preferential independence does not hold can often be circumvented once detected. This can be done either by eliminating all alternatives with levels of an attribute that fall below a required threshold, as by eliminating from consideration all graphics programs with insufficient resolution; or by reformulating the problem into one with attributes that are preferentially independent (see box).

Counterexample to Preferential Independence

Consider attributes of a diet measured in terms of the weight of each element of the diet:

X_1 = Steak X_3 = Bread

X_2 = Potatoes X_4 = Desserts

A person with enough money to have plenty of bread and desserts each day, ($X_3 = 5$, $X_4 = 6$) say, is likely to prefer a high-meat main dish with fewer calories to one with low meat and high calories. Thus,

(slices of meat, few potatoes) > (no meat, many potatoes)

when you have:

(much bread, much dessert)

If this person becomes poor, however, and cannot generally have enough calories, then the preferences would change. The high calorie, low-meat meal would now be preferred to the other:

(no meat, many potatoes) > (slices of meat, few potatoes)

when you have:

(no bread, no dessert)

this person is thus not preferentially independent over the attributes as defined.

The decision problem might be reformulated so that the person might be preferentially independent over a new set of attributes. This could be the case if the diet were expressed in terms of more fundamental elements such as calories, bulk, percent of fats, sugars, and so on.

Utility independence. This assumption concerns the intensity of preferences, not just their relative order as does preferential independence. It is a cardinal condition rather than an ordinal condition. It is thus much more demanding—and also much more useful—than preferential independence.

Specifically, the utility independence assumption is that the indifference between a lottery and a certainty equivalent for any attribute does not depend on the levels of the other attributes. This is a most valuable condition because it implies that it is possible to measure the way the utility changes over one dimension, independent of all other attributes. These independent measurements can then be combined to give the multiattribute utility function.

Formally, the assumption of utility independence is that:

- If for any given level of attributes X_j other than X_i, there is an indifferent statement between several levels of X_i:

$$X_i' \sim (X_i'', P; X_i''') \text{ for one set of } X_j, j \neq i$$

- then this indifference holds for all levels of X_j, $j \neq i$.

Symbolically, this can be written as (where / again indicates conditionality)

$$\text{If} \quad (X_i'/X_j) \sim (X_i''/X_j, P; X_i'''/X_j)$$
$$\text{implies} \quad (X_i'/X_j') \sim (X_i''/X_j', P; X_i'''/X_j')$$

then X_i is utility independent of X_j, $j \neq i$. (See box for an illustration of this definition.)

People are quite commonly utility independent among different attributes. This assumption does not noticeably limit the way the multiattribute utility function can define the value of interactions between attributes. Paradoxically, it is the assumption that causes the least difficulty for the analyst even though it is

Example of Utility Independence

Consider the evaluation of the computer graphics programs again. Assume that when considering GRAF-ITTY, a particular release with known levels of speed and resolution, a professor is indifferent between buying the program at the discount introductory price of \$450 or taking the risk with probability $P = 0.8$ of getting a review copy at no cost and probability $P = 0.2$ of having to pay the full price of \$600. Thus,

$$\$450 \sim (\$0, 0.8; \$600)$$

This statement of course defines a relationship between the utility of three levels of cost for the GRAF-ITTY program. Specifically,

$$U(\$450) = 0.8U(\$0) + 0.2U(\$600)$$

The professor would be utility independent for cost if the same statements applied for any other graphics program, with different levels of speed and resolution.

It is important to note that these kinds of indifference statements can hold regardless of the level of the other attributes. Thus even if a particular graphics program is poor and not worth \$450 to the professor, the statement can hold: the professor might be indifferent between having to pay \$450 for sure for the junky program (if it were required by a consulting client, say) and of having an 80% chance of getting it for free and a 20% chance of having to pay \$600.

more demanding analytically: A person who is preferentially independent over attributes is almost certainly utility independent.

Utility independence of an attribute is often interpreted as meaning that the *"shape" of the utility function* over that attribute is the same, independent of the level of the other attributes. This is a useful shorthand, but one that must be interpreted carefully. Most obviously the statement that two utility functions have the same "shape" in this context means that if it is "risk averse" for one set of the other attributes it is not "risk positive" for some other set of those attributes. More precisely, the statement that two utility functions have the same "shape" is to be interpreted as meaning that they are positive linear transformations of each other as defined in Section 18.6:

$$U'(X_i) = aU(X_i) \pm b \qquad a > 0$$

Since such transformations can change the vertical scale considerably while keeping the X_i scale constant, utility functions with the same "shape" may look quite different (see box).

Utility Functions with the Same "Shape"

Consider the professor's indifference statement for the cost of graphics programs, given in the previous box:

$$\$450 \sim (\$0, 0.8; \$600)$$

One interpretation of this statement is that, considering less cost to be preferable:

$$U(\$600) = 0 \qquad U(\$0) = 1.0$$
$$U(\$450) = 0.8$$

This utility function is sketched in Figure 20.3(*a*).

Many other utility functions for money match the above indifference statement. For example consider one in which $U(\$600)$ is raised by adding 0.5, and the vertical scale shrunk to 0.2 of what it is above:

$$U'(\text{cost}) = 0.2U(\text{cost}) + 0.5$$

Then we have:

$$U'(\$600) = 0.5 \qquad U'(\$0) = 0.7$$
$$U'(\$450) = 0.66$$

This utility function for cost, which might be appropriate for another program with different levels of speed and resolution, is sketched in Figure 20.3(*b*). It looks quite a bit different from the other, being much flatter; nonetheless it has the same "shape" within the meaning of the phrase used in the context of multiattribute utility.

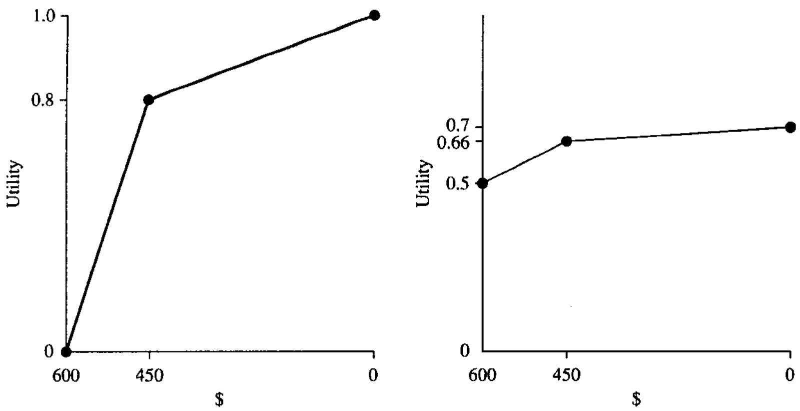

FIGURE 20.3
Two utility functions with the same "shape." Although visually dissimilar they both satisfy the same indifference statement: \$450 ~ (\$0, 0.8; \$600) so that $U(\$450) = 0.8U(\$0) + 0.2U(\$600)$.

Consequences. A practical method for measuring multiattribute utility is possible if both the preferential and utility independence assumptions are acceptable. This method both represents the value that people place in the interactions between attributes and is sufficiently efficient as regards calculations to be feasible in real situations.

When the preferential and utility independence assumptions hold, the multiattribute utility function $U(\mathbf{X})$ is defined by:

$$KU(\mathbf{X}) + 1 = \prod (Kk_iU(X_i) + 1)$$

where

- $U(\mathbf{X})$ is scaled from $U(\mathbf{X}_*) = 0$ to $U(\mathbf{X}^*) = 1.0$ when all X_i are respectively at their worst and best levels.
- $U(X_i)$ are likewise one-dimensional utility functions for each X_i, scaled so that $U(X_{i*}) = 0$ and $U(X_i^*) = 1.0$ for the worst and best levels of X_i.
- k_i are individual scaling factors for each dimension of attribute, measured in a way to be defined subsequently.
- K is a normalizing parameter that insures consistency between the definitions of $U(\mathbf{X})$ and the $U(X_i)$. It is commonly referred to as "big kay," to distinguish it from the k_i factors.

The crux of the equation for the N-dimensional multiattribute utility is that this function is defined entirely by

- N one-dimensional utility functions
- N scaling factors

This means that the number of points to be measured for any specified level of detail is a linear rather than an exponential function of the dimensions.

The computational advantage of this approach becomes apparent when it is compared to the requirements for measuring the multidimensional utility by the direct extension of utility theory discussed in Section 20.2. In that case, the number of calculations required is $(M^N - 2)$ where each $U_i(X_i)$ is defined by M points. Using the formula above, the number of points required is $(M - 1)$ for each dimension, being $(M - 2)$ for each utility function plus 1 for the scaling factor. The computational advantage of the multiattribute utility function is thus that:

$$N(M - 1) << (M^N - 2)$$

With this approach it is thus practical to measure six-dimensional utility functions; using the same level of detail as in Section 20.2, $M = 5$ and 24 points must be measured, well within the practical limits for measuring utility. We in fact measured six-dimensional utility functions for the government of Mexico to help them determine a strategy for developing the airport system for their capital.

The nature of the formula for multiattribute utility becomes more evident when we examine the two-dimensional case. We then can also obtain an explicit solution for both $U(\mathbf{X})$ and the normalizing factor K. For two dimensions then,

$$KU(X_1, X_2) + 1 = [Kk_1U(X_1) + 1][Kk_2U(X_2) + 1]$$

Multiplying this expression out and eliminating common terms thus gives:

$$U(X_1, X_2) = k_1U(X_1) + k_2U(X_2) + Kk_1k_2U(X_1)U(X_2)$$

We can thus see that the multiattribute utility is the weighted sum of the one-dimensional utilities, modified by terms accounting for the interaction between the attributes.

The one-dimensional utility functions, $U(X_i)$ are measured just as described in Section 19.5. In doing this we need not be concerned with the level of the other attributes because, with the assumption of utility independence, we know that the "shape" of each $U(X_i)$, is the same for all levels of the other attributes. The only new factor we need to pay attention to in the measurement of the $U(X_i)$ is to be sure to scale them so that $U(X_{i*}) = 0$ and $U(X_i^*) = 1.0$, to be consistent with the formula for multiattribute utility.

The normalizing factor K, that insures consistency between the definition of $U(\mathbf{X})$ and the $U(X_i)$, can be calculated from these definitions. Note carefully that by definition for the equation $U(\mathbf{X}^*) = 1$, and each $U(X_i^*) = 1$ when all the X_i are at their best levels. Substituting these values into the formula for multiattribute utility we obtain an implicit expression for K:

$$K + 1 = \prod (Kk_i + 1)$$

In general this expression, an $(n - 1)$-dimensional polynominal, must be solved for K by trial and error.

The determination of K is facilitated by the fact that its value is bounded by the sum of the k_i scaling factors for each dimension. Thus,

$$\sum k_i < 1.0 \qquad \text{implies} \qquad 0 < K$$

$$\sum k_i > 1.0 \qquad \text{implies} \qquad -1 < K < 0$$

For the special case when $\sum k_i = 1.0$, the solution for K degenerates and the multiattribute utility is the simple additive model:

$$\sum k_i = 1.0 \qquad U(\mathbf{X}) = \sum k_i U(X_i)$$

In practice, the k_i that one obtains from people rarely add up to 1.0. This fact underlines the limited value of the additive model, already stressed in Section 20.2.

For the two-dimensional case there is an explicit solution for K. Thus:

$$K + 1 = K^2 k_1 k_2 + K(k_1 + k_2) + 1$$

so that

$$K = (1 - k_1 - k_2)/k_1 k_2$$

The solutions for this expression are summarized in the box.

This solution in turn leads to a simple solution for the two-dimensional $U(\mathbf{X})$. By replacing the above solution for K in the formula, we obtain

$$U(X_1, X_2) = k_1 U(X_1) + k_2 U(X_2) + (1 - k_1 - k_2)\, U(X_1) U(X_2)$$

This formula clearly indicates how the interactive terms disappear when $\sum k_i = 1.0$, resulting in the exceptional additive case.

Finally, the scaling factors k_i need to be obtained by a special procedure. To appreciate this step, it is helpful to understand what the k_i represent. Each k_i is the multiattribute utility of the best level of its attribute i, when all the other attributes $X_j, j \neq i$, are at their worst levels. Figure 20.4 illustrates the situation in two dimensions. In general, these special combinations of (X_i^*, X_{j*}) are referred to as "corner points," by analogy to this figure.

The k_i are thus obtained with respect to the reference points for the multiattribute utility:

$$U(\mathbf{X}_*) = 0 \qquad U(\mathbf{X}^*) = 1.0$$

Solutions for K in Two Dimensions

k_1	$k_2 = 0.1$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	80	35	20	12.5	8	8	2.9	1.2	0
0.2		15	8.3	5	3	1.7	0.7	0	−0.55
0.3			4.4	2.5	1.33	0.55	0	−0.42	−0.74
0.4				1.25	0.5	0	−0.38	−0.63	−0.83
0.5					0	−0.33	−0.57	−0.75	−0.88
0.6	Symmetric					−0.56	−0.71	−0.83	−0.93
0.7							−0.82	−0.89	−0.95
0.8								−0.94	−0.97
0.9									−0.99

Note that

- For $k_1 + k_2 < 1.0$ $\qquad K > 0$
- For $k_1 + k_2 > 1.0$ $\qquad -1 < K < 0$
- For $k_1 + k_2 = 1$ $\qquad K = 0$ leading to the additive form as a special solution

$$U(X_1, X_2) = k_1 U(X_1) + k_2 U(X_2)$$

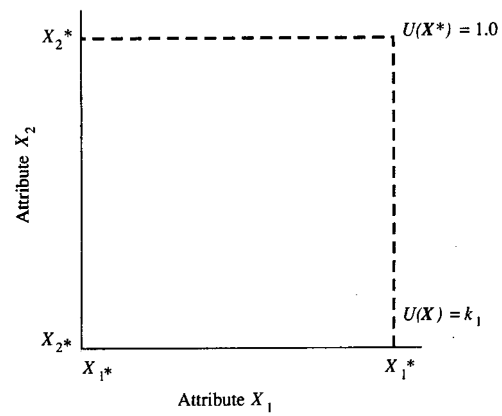

FIGURE 20.4
The scaling factors k_i represent the multiattribute utility of the corner points, where X_i is at its best level and all other $X_j, j \neq i$, are at their worst level.

Specifically, the object is to find an indifference statement that defines the multiattribute utility of each corner point:

$$(X_i^*, X_{j^*}) \sim [\mathbf{X}^*, P_i; \mathbf{X}_*]$$

So that

$$k_i = U(X_i^*, X_{j^*}) = P_i U(\mathbf{X}^*) + (1 - P_i)U(\mathbf{X}_*) = P_i$$

Since the indifference statement defines all possible levels of X_i, the only element that can be changed is P_i. The indifference statement is thus determined by the lottery equivalent/probability method described in Section 19.4.

20.4 MEASUREMENT

The measurement of a person's multiattribute utility function for a given situation is a straightforward implementation of the theory. It consists of the following steps:

- Specification of the Attributes
- Verification of Assumptions
- Measurement of One-dimensional Utilities $U(X_i)$
- Measurement of Scaling Factors, k_i
- Calculation of Normalizing Parameter, K
- Determination of Multiattribute Utility, $U(\mathbf{X})$

Each step is discussed in turn with reference to specifics given in boxes.

Specification of attributes. The analyst must first of all determine which are the most important attributes of a particular problem. This may require a certain amount of initiative and care. Many people are not used to thinking in many dimensions, and may not have already formulated their problem this way. The analyst should then help them identify the essential features—and thus attributes—of the problem (see box). Further, the analyst must consider and try to avoid the possibility that a person will not be preferentially independent for a specification of attributes, as in the diet example of the previous section.

Once the analyst and the client—an individual or an organization—agree on the attributes, they have to define the upper and lower ranges of interest for each attribute. They must also identify which level of each is the most and least preferred. This is as for one-dimensional utility.

Verification of assumptions. As people tend to be preferentially independent between attributes unless there is some particular kind of interaction between them—as in the diet preferences of the rich and poor given in the example of Section 20.3—the analyst should be on the watch for such possibilities. It is thus important for the analysts to understand the problem in some detail. If they do not, they may overlook fundamental difficulties, in particular that preferential independence is not acceptable for a certain situation.

The general procedure for verifying that the preferential independence assumption is acceptable for any two attributes is to ask the client

- to rank a few pairs of these two attributes.
- to ask if these rankings would be changed by variation in the levels of the other attributes, particularly changes that the analyst suspects might cause difficulties. If the answer is no, then the preferential independence assumption is acceptable for those two dimensions. The process should be repeated for each two pairs of attributes.

Specification of Attributes for Mexico City Problem

When we were approached by the government of Mexico to work on the decision about a possible second airport for Mexico City (as discussed in Section 16.3) our first task was to structure the problem and define the attributes of interest.

In thinking about the attributes, we had to recognize that this decision could have a major effect on the structure of the city and would thus affect groups with many different kinds of concerns. Three major groups seemed salient:

- The government, as promoter of the project
- The air travellers, as users
- The city residents, as bystanders affected by the airport

Our first suggestion to the government was that the attributes should reflect the essential features, and thus the main concerns of these groups. The attributes we selected for the analysis were then:

- "Airport Cost" and "Airport Capacity"
- "Safety" and "Access Time"
- "Noise" and "People Displaced"

These were acceptable to the government and we proceeded to measure six-dimensional utility functions.

It turned out that some of the attributes were not really significant. For example, the noise caused by aircraft was, at that time, of little concern in Mexico City, which was very noisy in any case due to buses and cars. In retrospect we should have chosen a different set of attributes which more accurately reflected the issues important to our client. As a general rule, any analyst working with a person or group should be alert to the desirability of changing the specification of the attributes to conform to the needs of the persons they are trying to help.

The procedure for checking the utility independence assumption is similar: the analyst asks the person whose utility is being assessed if the indifference statements being provided would be changed if the level of the other attributes were changed. Because the verification of the utility independence thus requires at least portions of the one-dimensional utility functions, it is frequently most convenient to check this assumption as part of the assessment of these utility functions.

Measurement of one-dimensional utilities. This is done in the standard ways, as described in Section 19.4. No change is necessary for the multiattribute case.

Measurement of scaling factors, k_i. The method for finding each scaling factor is to obtain an indifference statement of the form:

$$(X_i^*, X_{j*}) \sim (\mathbf{X}^*, P_i; \mathbf{X}_*) \qquad j \neq i$$

This indifference statement must be developed by varying the probability P_i, instead of the certainty equivalent (X_i^*, X_{j*}), which we want to keep fixed at the corner point.

The probability P_i is found by the Lottery Equivalent/Probability method (see Section 19.4) using bracketing (see Section 19.4). The analyst thus first proposes a high value of P_i and then a low one, iteratively narrowing the gap until the person whose multiattribute utility is being measured accepts a specific P_i. This is then the scaling factor k_i (see box).

Calculation of normalizing parameter K. The normalizing factor, K, is determined from the formula

$$K + 1 = \prod (Kk_i + 1)$$

except if $\sum k_i = 1.0$, as pointed out in Section 20.3. If we have only two dimensions, an explicit expression for K is available:

$$K = (1 - k_1 - k_2)/k_1k_2$$

For three or more dimensions, K must be determined iteratively. Analysts who have to do this frequently will develop a small computer routine to do so, using a standard search technique such as the Newton-Raphson method available in most large computer libraries.

Determination of multiattribute utility. The obvious way to obtain $U(\mathbf{X})$ is by using its formula

$$KU(\mathbf{X}) + 1 = \prod (Kk_iU(X_i) + 1)$$

This works well whenever the one-dimensional utilities $U(X_i)$ are defined accurately for each X_i that might be of interest.

Scaling Factors in Automobile Industry

In the automobile industry, designers can now choose between many different materials: a variety of alloys of steel, of aluminum, and a range of plastic composites. For many parts of the car, their concern focuses on the cost and the weight, the latter being important because it influences fuel economy.

In working with the industry, our group at MIT found that multiattribute utility functions over cost and weight are really helpful to the designers. Specifically, we develop utility functions for the range of attributes suitable for the particular part or system being designed, such as a bumper.

For a bumper, the weight might range from 20 to 50 kg, and the manufacturing cost from \$50 to \$150. Least amounts are of course preferred so that

$$\mathbf{X}_* = (50 \text{ kg}, \$150)$$

$$\mathbf{X}^* = (20 \text{ kg}, \$50)$$

To find the scaling factor for weight, we need to obtain

$$(X^*_{\text{weight}}, X_{\$*}) = (20 \text{ kg}, \$150) \sim (\mathbf{X}^*, P; \mathbf{X}_*)$$

We did this by asking designers questions such as:

> "Suppose you were engaged in a research and development program. You know you could certainly make bumpers out of Alloy X with characteristics (20 kg, \$150). You could also choose a process that has a probability P of giving you the wonder alloy you have been looking for ($\mathbf{X}^*$) or which, with probability $(1 - P)$, might leave you with unimproved current material ($\mathbf{X}_*$). Would you opt for Alloy X if P = 90%? 5%? . . . and so on."

This process works well. By putting the person being interviewed into a familiar, realistic context, it leads to an easy measurement of the scaling factors.

In practice we find that the k_i found by this kind of process do quite clearly reflect the relative weighting anyone would expect. For example, in considering the relative merits of industrial development and the environment, the "greens" clearly give a much higher k_i to the environment than industrialists.

In practice, it often occurs that we want $U(\mathbf{X})$ for combinations of X_i whose utility lies somewhere in between the points measured explicitly. It is then necessary to interpolate. This may be done either by formula or linearly.

If the interpolation is done by using standard formulas (see Section 19.5) of the form

$$U(X_i) = a + be^{-cXi} \quad \text{or} \quad = a + bX^c$$

Then these expressions can be directly introduced into the formula for the multiattribute utility.

Often, however, linear interpolations must be used. This would be because the one-dimensional utility functions are too irregular to fit a smooth formula. This requires a careful extension to the multidimensional utility.

The essential concept in applying linear interpolations to $U(\mathbf{X})$ is that the "shape" of the one-dimensional changes in utility—parallel to any X_i axis—remain the same. As defined in Section 20.3, this means that the relative difference in utility $U(X_i)$ between its two extreme points, $U(X_{i*})$ and $U(X_i^*)$ is constant throughout the domain of $U(\mathbf{X})$. To find the utility along any X_i dimension we therefore simply effect a positive linear transformation on the original $U(X_i)$, scaled from 0 to 1, of the form

$$U'(X_i) = aU(X_i) + b \qquad a > 0$$

In doing this it is most important to be careful about the definition of a and b.

Linear interpolation for $U(\mathbf{X})$ is almost exclusively done for two dimensions due to the complexity of visualizing higher order spaces. It is therefore explained and illustrated for two dimensions. The procedure is

- Assign the known utilities to the corner points of the space defined by the range of attributes. These are $U(\mathbf{X}_*) = 0$, $U(\mathbf{X}^*) = 1.0$ for the worst and the best levels of all attributes; and $U(X_i^*, X_{j*}) = k_i$ for each corner point.
- Interpolate one-dimensional utilities from the "origin" of $U(\mathbf{X})$, $\mathbf{X}_*$ where $U(\mathbf{X}_*) = 0$, to the corner points. In this case,

$$U(X_i, X_{j*}) = k_i U(X_i)$$

- Interpolate one-dimensional utilities from the corner points to the apex $\mathbf{X}^*$ where $U(\mathbf{X}^*) = 1.0$. The positive linear transformations are

$$U(X_i^*, X_j) = (1 - k_i)U(X_j) + k_i$$

  since we now must interpolate from a starting utility equal to that of the corner point, that is k_i, and scale the utility of the other dimension to the range available, $(1 - k_i)$, as shown in the box.
- Interpolate, on lines parallel to either the X_i or X_j axis, to obtain $U(\mathbf{X})$ for any point inside the range of the attributes, being careful to define the transformation so that the base and scale are correctly defined: the base, b, is the end of the line with the lowest $U(\mathbf{X})$, the scale is given by the difference between the lowest and the highest $U(\mathbf{X})$ on that line (see box).

Interpolation for Two-dimensional Multiattribute Utility

A designer in the automobile industry, responsible for bumpers, has the one-dimensional utility functions for weight and cost shown in Figure 20.5. For simplicity, we assume that the designer's utility for dollars is a straight line

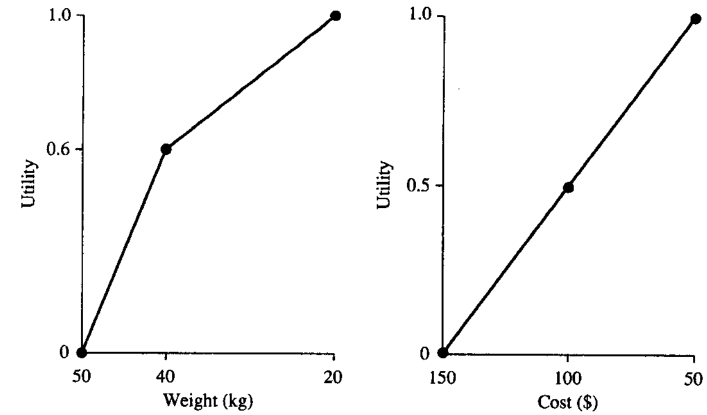

FIGURE 20.5
One-dimensional utilities used for interpolation of multiattribute utility function.

(perhaps because the amounts are small compared to the total cost). Suppose that we have already found that the designer's scaling factors for weight and cost are

$$k_w = 0.7 \qquad k_\$ = 0.4$$

The first step of the interpolation process is to assign the utilities to the four corners of the space defined by the range of attributes:

$$50 \text{ kg} \geq \text{weight} \geq 20 \text{ kg}$$
$$\$150 \geq \text{cost} \geq \$50$$

This is done in Figure 20.6.

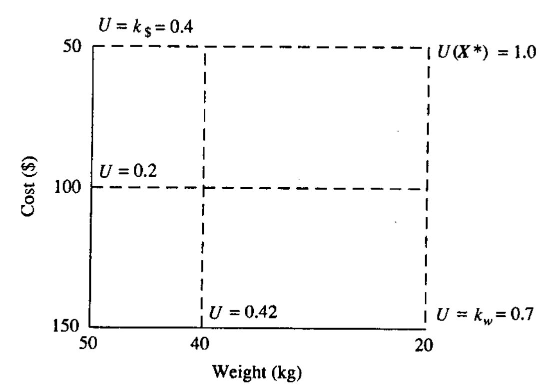

FIGURE 20.6
Utilities assigned after the first two steps of the interpolation process.

Now we interpolate the one-dimensional utilities along the axes from the "origin" of $U(\mathbf{X})$ at $\mathbf{X}_*$. For weight we thus have, always at the worst level of cost, $X_\$ = \$150$:

$$U(50,150) = 0 \qquad U(20,150) = 0.7$$
$$U(40,150) = 0.7(0.6) = 0.42$$

Likewise for cost we have, with weight at its worst level, $X_W = 50$ kg:

$$U(50,150) = 0 \qquad U(50,50) = 0.4$$
$$U(50,100) = 0.5(0.4) = 0.2$$

These numbers also appear in Figure 20.6.

The interpolation on the remaining sides now proceeds as above, except that we must remember that the linear transform of $U(X_i)$ involves a change in the base. Thus, along the top of the region, where $X_\$ = \$50$, we have

$$U(50,50) = 0.40 \qquad U(20,50) = 1.0$$
$$U(40,50) = (1 - 0.4)(0.6) + 0.4 = 0.76$$

Likewise for cost, along the right where $X_W = 20$ kg:

$$U(20,150) = 0.7 \qquad U(20,50) = 1.0$$
$$U(20,100) = (1 - 0.7)(0.5) + 0.7 = 0.85$$

These utilities are indicated in Figure 20.7.

Finally, the utility of points interior to the domain are interpolated from the knowledge of the utilities along the edge. The utility of point A in Figure 20.7 with $X_W = 40$ and $X_\$ = 100$ can thus be found by varying $X_\$$ along $X_W = 40$ kg:

$$U(40,150) = 0.42 \qquad U(40,50) = 0.76$$

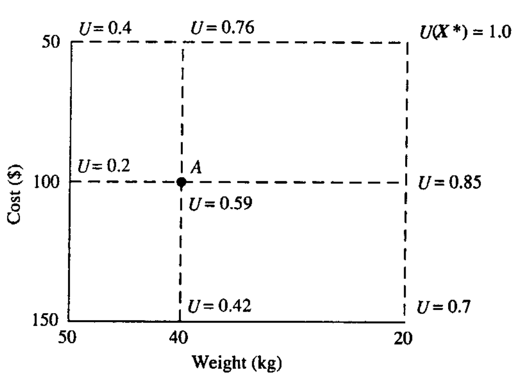

FIGURE 20.7
Utilities assigned by end of interpolation process.

$$U(40,100) = (0.76 - 0.42)(0.5) + 0.42 = 0.59$$

It can also be found by varying X_W, along $X_\$ = \$100$:

$$U(50,100) = 0.2 \qquad U(20,100) = 0.85$$
$$U(40,100) = (0.85 - 0.2)(0.6) + 0.2 = 0.59$$

20.5 APPLICATIONS

This section discusses two applications of multiattribute utility analysis, first to the Mexico City Airport and second to the selection of new materials. The Mexico City case is the classic: it was the first major application of the Keeney-Raiffa approach, it was done with those principals, and has been extensively reprinted. It also seems appropriate to bring closure to the many references made to it throughout the text. This discussion focuses on the actual results of the multiattribute utility analysis.

A set of applications to the evaluation of materials comes next. These represent the results of the latest work being done, and illustrate the range of uses to which this analysis can be put.

Mexico City airport. This case concerns the location of new airport facilities for Mexico City. The issue as presented to us by the government of Mexico was polarized around two locations, Texcoco and Zumpango. One faction wanted to keep all activity at the existing site, the "All Texcoco" alternative; the other faction wanted to move everything to a new facility, the "All Zumpango" solution.

As all such problems, the situation is really much more complex. Looking at the ways the different possible sizes of the facility combine with the different locations and can be phased over time, the number of logical possibilities was:

$$[(\text{Sizes})^{(\text{Locations})}]^{(\text{Periods})} = 4096$$

as indicated in Section 16.3. Further, as stated in Section 20.4, it seemed reasonable to consider six major attributes to the problem. We thus performed a decision analysis on this basis. The assessment of the multiattribute utility function was straightforward, even on this the first major application. The major difficulties were in fact not technical but concerned our mutual problems in translating between Spanish and English. In detail, we assessed utility functions for each of the major directors of the Ministry of Public Works. These were relatively homogenous, as they often are in a work unit that has grown to share common values, so that we felt comfortable in averaging their parameters to obtain a collective multiattribute utility function for the Ministry itself.

Integrating this function with the decision analysis we could obtain the optimal choices. Both of the salient choices, the "All Texcoco" and the "All Zumpango," ranked right near the top—as indeed they should have since these were preferred solutions.

The salient result of the multiattribute utility analysis was that it also ranked a wide range of compromise solutions right at the top. In fact, the difference in value between the top 10 to 20 choices were down in the second and third decimal places, way beyond the accuracy of any utility assessment. The multiattribute utility analysis made clear, by recognizing the trade-offs between attributes and incorporating them into the analysis, that there were many possible compromise choices that were acceptable. The common feature of each was that it provided insurance against risks and flexibility to adapt as necessary. As practitioners we have now come to recognize this discovery of compromise alternatives as a standard result of the use of multiattribute utility.

This analysis thus provided the effective basis for developing a strategy for development for Mexico City's airports. This was much along the lines presented for Sydney's Airports in Sections 16.8 and 16.9. The difference between that application and the earlier one for Mexico City is that, having learned that multiattribute utility systematically identifies compromise solutions, and knowing what kind of insurance makes sense for airports, we could skip the multiattribute utility analysis for the second situation, a virtual replay of the Mexico City case.

Materials selection. The choice of materials has become a considerable problem for engineers. The number and type of materials available has grown rapidly in recent years, and vastly expanded the possibilities that a designer should consider seriously. In designing cars for example, one only used to have to consider several grades of rolled steel; now it is realistic to evaluate also the potential of other materials, such as aluminum, powdered metals, and an extraordinary range of plastics, each having quite different characteristics. Materials selection is now truly a multiattribute utility problem.

The Materials Systems Laboratory at MIT has been developing the many different ways multiattribute utility analysis can be useful to designers. By doing this in close collaboration with major international producers and users of materials, they have been able to document which applications are truly practical.

The most obvious application is to the selection of the preferred designs for a part or an assembly. This implies a choice of materials in fact since different designs will be best executed with different materials. For example, one automobile bumper may involve honeycomb plastics to absorb energy, another a solid steel beam, and a third structural thermoplastics.

In practice, the decision analysts will assess the multiattribute utilities of the members of the design team, and apply them to the possible choices. If the designers have relatively similar utility functions, this process will lead them to the design that offers them the best combination of their desired characteristics. Often this choice will be a compromise—as in the case of the airport applications—between initially polarized views. Typically, the design that is ultimately favored was far from obvious at the start, so that multiattribute utility analysis makes a significant contribution.

When the designers turn out not to share the same preferences, the analysis will identify the fundamental reasons for their disagreements about design. We have found that this process can be most helpful in design groups. It clarifies their objectives and permits a clearer understanding of choices to be made.

A second application concerns research management. Major laboratories can estimate the kinds of achievements they might obtain from a given effort in various directions. The management task is to identify which lines of work give the highest value in terms of all the attributes of interest. Should the lab focus on one material that could significantly reduce weight at some cost and loss of its capability to reduce dents; or on another that promises cost reductions and extended service life, for some increase in weight? The analysis focuses on ranking the alternatives.

Another significant area of application concerns the producers of materials, who must constantly decide whether to introduce a new material and, if so, at what price. To resolve this question, producers obtain their customers' multiattribute utility functions. (Realistically, this may have to be done indirectly through independent consultants, to avoid the potential bias buyers may introduce when talking to sellers.) The analysis then constructs an isovalue line (see Section 18.4) through the properties of the material currently used for some application, as in Figure 20.8. They can use this information to decide what to do.

We have observed several instances in which the analysis demonstrated that a new, technically superior material simply could not be produced at a competitive price. Graphically, its minimum cost (at A) was too high; its advantages in lower weight, for example, did not compensate for the extra costs required to produce it. In another instance, the analysis permitted a producer to estimate how high one could price a new material so that, at B, it would be about as high as it could be while still offering greater utility to the potential users of the material.

Overall, the application of multiattribute utility analysis provides, for really the first time, a rational method of evaluating materials which fully incorporates the many simultaneous, nonlinear trade-offs designers do make among the characteristics of a system. The experience in materials selection can and has been extended to many other fields.

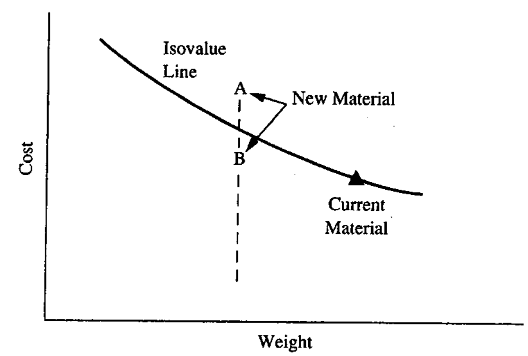

FIGURE 20.8
Use of multiattribute utility analysis in decisions about introducing and pricing new materials.

REFERENCES

de Neufville, R., and R. Keeney, (1972, 1976). "Use of Decision Analysis in Airport Development in Mexico City," Chapter 23, *Analysis of Public Systems,* A. W. Drake, R. Keeney, P. Morse, eds., MIT Press, Cambridge, MA; Chapter 24, *Systems Planning and Design,* R. de Neufville and D. H. Marks, eds., Prentice-Hall, Englewood Cliffs, NJ.

Field, F., and R. de Neufville, (1988). "Materials Selection—Maximizing Overall Utility," *Metals and Materials,* May, pp. 378–382.

Keeney, R., (1978). "Utility Independence and Preference for Multiattributed Consequences," *Operations Research,* Vol. 19, pp. 875–893.

Keeney, R., and Raiffa, H., (1976). *Decisions with Multiple Objectives,* Wiley, New York.

PROBLEMS

20.1. *General I*

Given $U_x(X) = \frac{1}{32}(X^2 - 4) \qquad 2 \le X \le 6$

$$U_y(Y) = \frac{1}{6}(Y + 36)^{1/2} \qquad -36 \le Y \le 0$$

and $k_x = 0.5 \qquad k_y = 0.6$

(*a*) Give the value of $U(X,Y)$ for all four corner points for which the multiattribute utility is defined.
(*b*) Find by interpolation $U(4, -36)$; $U(4,0)$; $U(4, -20)$.
(*c*) Write the general formula for $U(X,Y)$.
(*d*) Solve for the value of K in the formula.
(*e*) Calculate $U(4, -20)$ by using the formula.

20.2. *General II*

Given $U_x(X) = \frac{X(X + 6)}{160} \qquad 0 \le X \le 10$

$$U_y(Y) = \frac{(Y^{1/2} - 5)}{5} \qquad 25 \le Y \le 100$$

and the following indifference statements:

$$(10, 25) \sim [(10, 100), 0.6; (0, 25)]$$

$$(0, 100) \sim [(10, 100), 0.6; (0, 25)]$$

(*a*) Give the value of $U(X,Y)$ for all four corner points for which the multiattribute utility is defined.
(*b*) Find by interpolation $U(4,25)$; $U(4, 100)$.
(*c*) Write the general formula for $U(X,Y)$.
(*d*) Solve for the value of K in the formula.
(*e*) Calculate $U(4,100)$ by using the formula.

20.3. *Workstations, again*
The original interviews (see Problem 19.8) also revealed that, when confronted with the possibility, Lee was indifferent between the low price, slow model, and a strategy which led to an even chance of getting the best or worst combinations.

But when she could have the high price, fast model, a strategy with only a 30% chance of getting the best would be just as acceptable.

$$(\$1000, 4) \sim (\text{Best}, 0.5; \text{Worst})$$

$$(\$6000, 24) \sim (\text{Best}, 0.3; \text{Worst})$$

(*a*) Identify the best and worst cases in terms of ($, Speed) pairs.
(*b*) Sketch the (Cost, Speed) plane indicating the utility values you can identify from the preceeding information.
(*c*) Using the information from previous problems, what is the utility of Workstation A = ($4000, 24)? Workstation B = ($1000, 10)?
(*d*) State the formula for the two-dimensional utility and use it to obtain the utility of Workstation C = ($4000, 18).
(*e*) Help Lee by ranking the workstations from most to least desirable.

20.4. *Basketball & Movies*
Bobby West likes to spend his leisure time playing basketball or going to movies. His utility functions for basketball games and movies per month are given in the tables below.

Basketball games per month	One-dimensional utility
0	0
2	0.1
4	0.6
6	0.9
8	1.0

Movies per month	One-dimensional utility
0	0
1	0.2
3	0.6
5	0.9
8	1.0

Bobby has expressed the following indifference statements:

$$(8B, 0M) \sim [(8B, 8M), 0.7; (0B, 0M), 0.3]$$

$$(0B, 8M) \sim [(8B, 8M), 0.4; (0B, 0M), 0.6]$$

(*a*) Draw a graph with axes labeled basketball games per month and movies per month. Indicate multiattribute utilities at the four corner points.
(*b*) Define and explain the significance of utility independence.
(*c*) Assuming utility independence, calculate the multiattribute utility of the following points by interpolation and label them on the graph from (*a*):

$$U(4B, 0M) \qquad U(0B, 5M) \qquad U(4B, 8M) \qquad U(8B, 5M)$$

(*d*) Based on your results in (*c*), interpolate to find U(4B, 5M) and label this on the graph from (*a*).
(*e*) Write the general formula $U(X, Y)$.
(*f*) Solve the formula for U(4B,5M) and compare answer to your results in (*d*).

20.5. *C. Eric Lubb*
C. Eric Lubb, environmental lobbyist, has utility functions for birds and trees as shown in the table.

Item	Number of species	Utility (one-dimensional)
Birds	0	0
	5	0.5
	15	0.75
	40	1.0
Trees	0	0
	5	0.1
	10	0.3
	15	0.75
	30	1.0

(*a*) In considering the Rat Race National Park, the Congress seems to be headed toward approval of a Pesticide Treatment Plan (PTP) that will allow a total of 30 tree species to be established, at the expense of killing off all of the birds. Eric is undecided whether to try to get the bill back into committee, which has a 30% chance of approving the Environmentally Sound Management Plan (ESMP), which would give the park 40 kinds of birds and 30 of trees, but a 70% chance of allowing strip mining (no birds, no trees). Express as an indifference statement.

(*b*) In some consistent way, assign values of *U*(PTP), and *U*(ESMP), and *U*(strip mining).

(*c*) Eric says "That blasted Pesticide Plan is as bad as having 15 bird species and no trees." What is *U*(15 birds, 0 trees)?

(*d*) What assumptions must you make to facilitate the construction of a utility function from the information given? What sort of questions would you ask to test these assumptions?

(*e*) Sketch *U*(0, T), *U*(15, T), and *U*(40, T) as functions of T.

(*f*) Would you expect Eric to prefer (15,10) or [(15, 30), 1/3; (0, 15)]?

(*g*) From the formula: $KU(B,T) + 1 = (Kk_t U_t(T) + 1)(Kk_b U_b(B) + 1)$ find an analytical expression for U(B, T) as a function of $U_b(B)$ and $U_t(T)$ and obtain *U*(15, 15) and *U*(5, 5).

20.6. *Auto Bumper*

Sloane Iacocca, an automobile executive, must select an appropriate material for an automobile bumper. The cost and weight are the criticial features of this problem. According to Sloane, "In order to stay within our EPA weight class, the bumper cannot weigh more than 50 pounds. Naturally, in the interests of fuel economy, we would like to keep the weight of the bumper down. However, as the weight of our feasible designs drops, the cost rises."

You are able to determine that Sloane is indifferent between:

- A 25 lb bumper and the lottery (50 lbs, 1/2; 15 lbs)
- A 35 lb bumper and the lottery (50 lbs, 1/2; 25 lbs)
- A bumper costing $60 and the lottery ($40, 1/2; $100)
- A $50 bumper and the lottery ($40, 1/2; $60)

(*a*) Fill in the following tables:

Bumper utilities—Sloane Iacocca

Cost ($)	*U*(cost)	Weight (lbs)	*U*(weight)
40		15	
50		25	
60		35	
100		50	

You now determine that Sloane is also indifferent between:

- A bumper that weights 15 lbs and costs $100 and the lottery [(15 lbs, $40), 4/10; (50 lbs, $100)].
- A bumper weighing 25 lbs and costing $40 is as valuable as the lottery [(15 lbs, $40), 3/4; (50 lbs, $100)].

(*b*) What assumptions must you make to construct Sloane's multiattribute utility function?

(*c*) What is *U*(15 lbs, $100), *U*(25 lbs, $40), *U*(15 lbs, $40), and *U*(50 lbs, $100)?

(*d*) What is *U*(50 lbs, $40)?

(*e*) What is the value of *K*?

(*f*) Assuming that Sloane is consistent, would she prefer bumper A (35 lbs, $50) or B (25 lbs, $60)?

20.7. *Developer*

A developer reviewing the design of a $20M building faces a tradeoff between construction costs and energy consumption. The current design requires 10 million btu/yr of heating and cooling power. Many modifications such as the addition of awnings, improved insulation, or solar panels could save up to 7 billion btu/yr, at a cost of $4M. Since the range of possibilities and the uncertainties are rather confusing, she wants you to help sort out her preferences.

From interviews you determine that, for building costs alone, she is indifferent between

$22.5M and ($24M, 0.5; $20M)
$23.4M and ($24M, 0.5; $22.5M)
$21.5M and ($22.5M, 0.5; $20M)

regardless of the level of energy used. Similarly, for energy measured in millions of btu/yr, she is indifferent between

5 and (10, 0.5; 3)
3.75 and (5, 0.5; 3)
7 and (10, 0.5; 5)

regardless of the level of costs. For combinations of costs and energy, she is indifferent between

($20M, 10) = the current plan
and [($24M, 10), 0.2; ($20M, 3)]

and also

($24M, 3) = maximum energy conservation
and [($24M, 10), 0.3; ($20M, 3)]

(*a*) Solve the indifference statements for their implied utility.
(*b*) Determine which plan the developer is apt to prefer:

Plan	Cost ($M)	Energy use
Current design	20	10
High insulation	21	7
Solar heating	22.5	5
Sheltered windows	23	4
Maximum conservation	24	3

(*c*) Would the developer prefer the current design or the alternative [(22.5, 7), 0.6; (22.4, 3.75)]?
(*d*) Find *K* by applying the general formula and check the results from part (*b*).

20.8. *Earthquake Code*
A research team investigating the desirable level of earthquake protection for apartment buildings obtains the following information from an apartment dweller:

- If the rent control board were reviewing the rent on her apartment, with equal chances of setting rents of $100 or $250, she would settle with her landlord for any monthly rent up to $200. If she thought she had an 80% chance of getting the $100 rent from the board, she would not settle for more than $150.
- Given the choice between one building code that would result in *X* earthquake deaths per 100,000 residents over the next 10 years, and another equally likely to result in 0 or 70 deaths per 100,000 residents, she prefers the fixed code for $X < 40$.
- If she were apartment hunting, expecting to find either a most desirable apartment ($100/month, 0 lives lost/100,000) or a least desirable apartment ($250, 70) depending on her luck, she would accept a cheap-but-risky compromise apartment ($100, 70), if her probability of getting the most desirable is less than 0.5. She would also accept an expensive-but-safe compromise apartment ($250, 0) if the probability of finding a most desirable apartment fell below 0.4.

(*a*) On a cost vs. lives graph, plot the utilities directly identified by the information given. Assume utility independence.
(*b*) Assuming linear interpolation, comment on the apparent desirability to our subject of the building codes with the following outcomes:

A. ($125, 70)   B. ($150, 40)   C. ($200, 20)

D. [($100, 70), *P*; ($225, 0)] as a function of P.

(*c*) Use the general formula to determine *K*.

(*d*) Use *K* to find *U*($150, 40) and *U*($200, 20). Check against your answers in (*b*).

20.9. *Chip Wright*
Chip Wright frequently flies between Boston and St. Louis, a trip that can take as little as 3 hours on a direct flight, or otherwise up to 5 1/2 hours. Chip likes to round up poker games. As he plays against strangers, he is never sure how well he might do, although he does expect a less conservative player out of Chicago than out of Boston. He has difficulty deciding which flight to take.

(*a*) In talking to Chip, he tells you that he once, during a 3 hour flight to St. Louis, paid a dollar to drop out of a game on which he had a 50-50 chance of winning $10 and losing $5. Sketch his utility function over money. Does Chip really seem like a gambler to you? Why?
(*b*) You also find that he just came home on a 4 hour flight from St. Louis, which he chose in preference to an even chance to make either the quickest flight back by placing himself on the waiting list or to have to catch a backup flight that would arrive 2 1/2 hours later. Anyway, he felt certain to lose $5 because this was his unlucky day. Sketch Charlie's utility function over travel time.
(*c*) You find, in the end, that Chip is indifferent between the following situations, given in terms of (time, money earned):

$$(4, -1) \sim [(4, -5), 0.6;\ (3, -1)]$$
$$(4, -1) \sim [(3, -5), 0.8;\ (5.5, -5)]$$
$$(4, -1) \sim [(4, 10), 0.5;\ (4, -5)]$$

How many indifference statements are necessary to scale the two-dimensional utility function? Which of the preceeding provide(s) the necessary information?
(*d*) Calculate *U*(5.5, 10) on a scale with *U*(3, −5) = 0 and *U*(3, 10) = 1.
(*e*) Sketch isoutility lines for *U* = 0.5, 0, − 1.

20.10. *Highway Project*
A highway project will have two serious effects on a town: commuter time savings and residential displacement. Their range appears to be

commuter time savings: $0 \le S \le 300$ hrs/day

residential displacements: $0 \le D \le 140$ families

You find that:

- In comparing the null alternative, ($S = 0$; $D = 0$), to a hypothetical alternative that has probability *P* of yielding (300,0) and probability $(1 - P)$ of producing (0, 140), The mayor seems to be indifferent between the two when $P = 0.7$.
- When choosing between (300, 140), the maximum impact project, and the hypothetical project the mayor seems to be indifferent between the alternatives when $P = 0.6$.
- Ignoring the level of D, the mayor is indifferent between $S = 100$ and a 50/50 lottery between $S = 0$ and $S = 300$.
- The mayor is indifferent between $D = 80$ and the lottery (0, 0.5; 140).

(*a*) What is the utility of (0, 0)?
(*b*) What is $U(300, 140)$?
(*c*) Assuming linear interpolations between assessed points, sketch $U(S, 140)$ and $U(0, D)$.
(*d*) Assuming mutual utility independence, sketch $U(S, 0)$. What is the utility of (100,0)?
(*e*) Sketch $U(100, D)$. What is the utility of (100, 80)?
(*f*) If the mayor is offered a choice of (100,80) or the lottery [(0, 80), 0.5; (200, 0)], which choice would you expect?

20.11. *Muckraker*
The editor of Exposé Publications is discussing with an agent a book that claims to be a true, behind-the-scenes view of State A and M.

"Although we have a 90% chance of netting $\$10^6$ from this book, there's a 10% chance that SAM could sue us while we only recover costs. While a little publicity is good for business, a SAM lawsuit would be a biggie (versus a small or average), which would be more than we can handle. I'm afraid your price is too high."

"But," replies the agent, "I know you took a chance on a biggie lawsuit last month."

"Yes, but we talked to the lawyers and had our 10% chance of a biggie (with a 90% chance of no suit) changed to a guarantee of a small suit. That was some negotiation session. Had the chance of the biggie been 20%, we would have taken an average suit instead."

"Okay," says the agent, "I see your point. But the money here is very good. I know you were between a 25% chance of netting 10^6 and a 75% chance of netting 0 last month and you took a guaranteed 2×10^5 overseas licensing contract instead. Surely a 90% shot at 10^6, lawsuit or no, is worth it.

"Let me think it over. I'll call you tomorrow."

(*a*) Describe as a lottery the choice facing the editor.
(*b*) Formulate the lotteries that describe the editor's attitude toward lawsuit size.
(*c*) What is *U* (lawsuit size)?
(*d*) What is the editor's utility function for net return on a book? Be sure to write the lottery.

In reviewing past business decisions, the editor finds two similar ones that can be described as:

$$(10^6, \text{Biggie}) \sim [(10^6, \text{no suit}), 0.96; (0, \text{Biggie})]$$

$$(0, \text{no suit}) \sim [(10^6, \text{no suit}), 0.8; (0, \text{Biggie})]$$

(*e*) Calculate the editor's utility function and prepare a table of values for all computable points of *U* ($, suit size).

20.12. *Great Western Children's Museum*
The Great Western Children's Museum is building new quarters, and the trustees must decide how much earthquake protection should be built into the structure. If there is a quake and the museum collapses with children inside, the trustees will be held responsible. The best estimates the trustees have been able to obtain have set the probability of a major earthquake in the next 20 years (the design horizon for the museum) as 40%. Three basic design options have been proposed:

Design	Extra cost	Probability of *L* deaths, given major quake			
		$L=0$	$L=30$	$L=50$	$L=100$
Standard	0	0.1	0	0.3	0.4
Super-rigid	$2M	0.4	0.5	0.1	0
Flexible	$5M	0.7	0.25	0.05	0

In negotiations the board of trustees has indicated the following indifferences:

For Lives: $75 \sim (100, 0.5;\ 0)$; $50 \sim (50, 0.5;\ 0)$; $30 \sim (50, 0.5; 0)$

For Extra Building Costs: $3M ~ ($5M, 0.5; 0); $2M ~ ($3M, 0.5; 0)

For Combinations of Lives and Costs:

(0, $5M) ~ [(100, $5M), 0.6; (0, $0)]   (100, $0) ~ [(100, $5M), 0.2; (100, $0)]

(*a*) Sketch the board's utility functions for lives and for costs.
(*b*) Formulate the two-dimensional utility function over lives and cost.
(*c*) Set up decision tree and determine optimal choice.

20.14. *Work and Play*
Here you are, looking for a job. Assume that, to a first approximation, your two dominant criteria are salary and days of vacation (S, V). You feel you really ought to get at least $30,000/year, to justify the expense of education. As regards vacation, you feel you need some at least, and would really like to have a lot so you could travel. Your one-dimensional utility functions thus are:

Salary ($K)	*U*(S)
20	0
25	0.1
30	0.5
35	0.8
40	0.9
50	1.0

Weeks off	*U*(V)
0	0
1	0.3
2	0.4
3	0.5
4	0.8
5	1.0

You find that you are indifferent between the job as a lifeguard in a holiday resort, (S, V) = (20, 5), and a 50:50 chance that you will get that super job (50, 5) or end up at the dreary grind (20, 0). On the other hand, you are also indifferent between the offer of a high-priced grind (50, 0) and a 50:50 chance between the lifeguard and the super job.
(*a*) By formula, determine whether you prefer (35, 2) or (30, 4).
(*b*) Verify (*a*) by interpolation, showing all work.
(*c*) Sketch isovalue lines on the S, V plane.