Flaws of Averages

--- Day 2: Teacher's Notes ---

With the video lesson shown and discussed yesterday, students have learned about three flaws of averages:

1) The average is not always a good description of the actual situation.
2) The function of the average is not always the same as the average of the function.
3) The average depends on your perspective.

To convey these concepts, the students were presented with the three real-world examples. The total length of the four in-class video segments was 12 minutes, leaving lots of time yesterday in your class session for you to work with the students on their own learning examples to firm up the ideas presented here on the flaws of averages.

Today, on Day 2, we go into some more detail, with more examples and more involvement with your students.

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The average is not always a good description of the actual situation.

1. Remember: Average water depth is one meter, and Dan can't swim!. Can the students think of other similar examples? Such as average slope of a hill for bicyclists; average number of meters above ground for a fast zip line; average number of cars parked in the parking lot of a shopping mall; etc. (class discussion).
2. Recall the song and dance towards the end of the BLOSSOMS video: The musical tone was the average of the real song, and the tap dance was one constant time between taps. Music and the arts – it's the non-averages that give joy and entertainment (class discussion).
3. Your diet – imagine eating your average diet each and every day. As with music, *"Variety is the spice of life!"*
4. "When Bill Gates enters a conference room, on average, everyone suddenly is a multi-millionaire." Is that useful or funny?
5. Three cars are parked next to each other. Two have 8-cylinder engines. The other has a 6-cylinder engine. What is the average number of engine cylinders per car? (ANSWER: (8 + 8 + 6)/3 = 22/3 = 7 1/3 cylinders.) An impossible outcome in reality. Not a good description of the actual situation.
6. Class: Others?

The function of the average is not always the same as the average of the function. (This is the most advanced concept from the video, requiring time for students to understand. But it's important!)

Remember the cookie example. Two cookies on each plate. Dan tells Rhonda that

1. Plate A has two cookies that have an average diameter of 7 cm.
2. Plate B has two cookies that have an average diameter of 8 cm.

Dan then asks Rhonda: "Which plate would you like to have if your goal is to have the largest cookie **area**?" That is, which do you choose if you want the eat the "most cookie"?

Towards Understanding. I don't know about you, but I love cookies! So, I'm replacing "diameter" with "radius" in the problem definition, just so we can have really large cookies! We could enjoy cookies with average radius of 8 cm or diameter 16 cm! Now that's a big cookie!

First, let's see what our intuition might tell us:

1. **INTUITION:** Remember that the area of a cookie is $A = \pi r^2$, where r is the cookie's radius. Note the nonlinearity: The area of the cookie increases as the square of the radius. With the cookies having different radii, as the radius of the smaller cookie is reduced by 1cm, the radius of the larger one is increased by 1 cm (to keep the average radius constant), and the increase in the radius of the 2nd cookie increases that cookie's area more than the decrease in radius of the first cookie decreases its area. It's all due to nonlinearity. So, we can keep the average radius fixed at a pre-specified constant, then decrease the radius of one cookie, increase the radius of the other by the same amount -- thereby keeping the average radius fixed --, and we are each time increasing the total area of the two cookies. Thus, Dan can trick Rhonda! Whew!

2. **PICTURES ARE WORTH THOUSANDS OF WORDS!** The above intuition is visually demonstrated in the set of 18 PowerPoint slides ("Cookies") on this website. Please slowly view the slides in Slide Show mode, starting with Slide #1, since there are animations on some of the slides and transitions between them.

3. **SO, YOU WANT SOME MATH?** We are looking for the average area of a random cookie in each of the plates A and B. If cookie #1 has radius r_1 and cookie #2 has radius r_2, then the correct total area of the two cookies is $\pi(r_1^2 + r_2^2)$. And the average area per cookie is $\pi(r_1^2 + r_2^2)/2$. (As mentioned above, the problem statement uses diameters, but we can easily switch back and forth, since the diameter d equals twice the radius r.)

   a) What is "the function"? The function is "area A of a cookie having radius r" = $A(r) = \pi r^2$.

b) In our problem, the average of the function is $[A(r_1) + A(r_2)]/2 = \pi(r_1^2 + r_2^2)/2$. This is the correct answer.

c) But 'the function of the average' means 'the function evaluated at the average radius,' or $A([r_1 + r_2]/2) = (\pi/4)(r_1^2 + 2[r_1 + r_2] + r_2^2)$.

d) These two formulas look quite different. Suppose the two cookies are identical, meaning the two radii are the same, $r_1 = r_2 = r$. Then, the average of the function is $[A(r_1) + A(r_2)]/2 = \pi(r_1^2 + r_2^2)/2 = \pi r^2$ and the function evaluated at the average radius = $(\pi/4)(r^2 + 2[r + r] + r^2) = (\pi/4)(r^2 + 4r + r^2) = (\pi)([r/2]^2 + r + [r/2]^2) = (\pi)(r/2 + r/2)^2 = \pi r^2$. The results are identical! But you can prove that this is the only case in which the results are identical. For proof details, see Supplement #2 of Flaws of Averages: https://blossoms.mit.edu/sites/default/files/video/download/averages-supplement-2.pdf

4. **LESSON WE HAVE LEARNED:** Be very careful when using a function. Do not put in the argument the average of two or more quantities. Rather, calculate the function's value for each of the possible values of the argument, and then average appropriately. Do that, and avoid a major Flaw of Averages! *Happy averaging to all!*

Wind Turbines.

Suggest you set up in class the advanced problem of location wind turbines for generating electricity https://blossoms.mit.edu/sites/default/files/video/download/averages-supplement-2.pdf, and mentioning in class the surprising conclusion: ***The best location of wind turbines is not necessarily the location with maximum average wind speed.*** Then have them read the solution at home. Discuss briefly the next day.

The average depends on your perspective.

1. First, see the examples worked out in the supplement: https://blossoms.mit.edu/sites/default/files/video/download/averages-supplement-3.pdf
2. Movie Madness - Ask students to estimate their answer to this question: *Of all the seats offered for sale in your local movie theater, what fraction do you think are actually sold to paying customers during an average week?* Collect their answers, have a student compute the average and write it on the blackboard. Typical answer: 80% of seats will be sold. But movie management reports that during a typical week only about 5% of offered seats are sold! (True!) How to explain the difference? ANSWER: We movie goers tend to go on Friday and Saturday nights, when almost all seats are sold. But virtually no one goes to the 11:00 AM Tuesday showing. Management laments 95% of seats not sold. Students who go only on Friday and Saturday night lament that sometimes their shows are sold out! Both views are correct, depending on your perspective.
3. Airlines: How might the Movie Madness logic apply to airlines? (Answer: Airline passengers are equivalent to the "movie goers" and airline management is equivalent to movie management.)