In the upcoming fractals lesson, we'll look at some mathematical ways to generate very interesting sets. In order to do this, we'll need to be able to work with the set of *complex numbers*, $\mathbb{C}$, which are an extension of the real numbers, $\mathbb{R}$ (those that can be written in decimal notation). [1]

This tutorial contains five exercises for you to practice with, and we've left enough space for you to work the problems right on these sheets. For clarity, sometimes use the $\times$ symbol to indicate multiplication (e.g. $x \times y$), and sometimes we'll just write the two numbers next to each other (e.g., xy); these two operations are the same.

1 Why Complex Numbers

Why do we need new numbers?

The hardest thing about working with complex numbers is understanding why you might want to. Before introducing complex numbers, let's back up and look at simpler examples of the need to deal with new numbers.

If you are like most people, initially number meant whole number, $0,1,2,3,\ldots$. Whole numbers make sense. They provide a way to answer questions of the form "How many ... ?" You also learned about the operations of addition and subtraction, and you found that while subtraction is a perfectly good operation, some subtraction problems, like $3-5$, don't have answers if we only work with whole numbers. Then you find that if you are willing to work with integers, $\ldots,-2,-1,0,1,2,\ldots$, then all subtraction problems do have answers! Furthermore, by considering examples such as temperature scales, you see that negative numbers often make sense.

Now that we have "fixed" subtraction, we will deal with division. Some, in fact most, division problems do not have answers that are integers. For example, $3 \div 2$ is not an integer. We need new numbers! Now we have rational numbers (fractions).

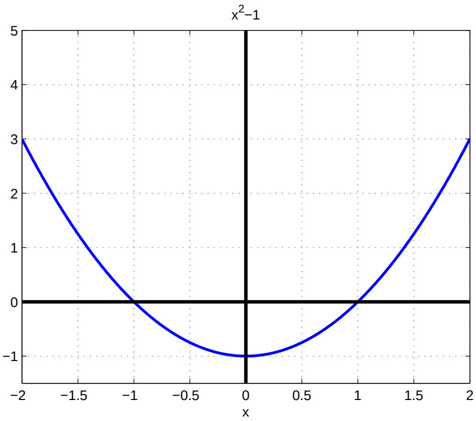
There is more to this story. There are problems with square roots and other operations, but we will not get into that here. The point is that you have had to expand your idea of number on several occasions, and now we are going to do that again.

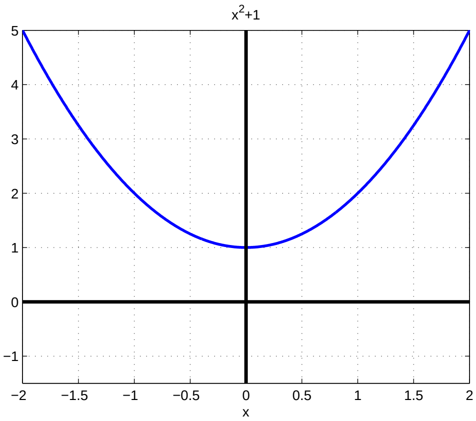
The "problem" that leads to complex numbers concerns solutions of equations.

- Equation 1: $x^2-1=0$. Equation 1 has two solutions, $x=-1$ and $x=1$. We know that solving an equation in x is equivalent to finding the x-intercepts of a graph; and, the graph of $y=x^2-1$ crosses the x-axis at $(-1,0)$ and $(1,0)$.

[1] Sections 1-4 are adapted with permission from Jeff Brown's course notes for Math 111 at University of North Carolina Wilmington, and can be found at `http://www.uncwil.edu/courses/mat111hb/Izs/complex/complex.html`.

- Equation 2: $x^2 + 1 = 0$. By the same logic, Equation 2 has no solutions; we can see this by looking at the graph of $y = x^2 + 1$.





Since the graph has no x-intercepts, the equation has no solutions. When we define complex numbers, Equation 2 will have two solutions.

2 The Number i

Consider Equations 1 and 2 again.

Equation 1	**Equation 2**
$x^2 - 1 = 0$	$x^2 + 1 = 0$
$x^2 = 1$	$x^2 = -1$

Equation 1 has solutions because the number 1 has two square roots, 1 and -1. Equation 2 has no solutions because -1 does not have a square root. In other words, there is no number such that if we multiply it by itself we get -1. If Equation 2 is to be given solutions, then we must create a square root of -1.

Definition: The imaginary unit i is defined by

$$i = \sqrt{-1}.$$

The definition of i tells us that $i^2 = -1$. We can use this fact to find other powers of i.

Example 1.

$$i^3 = i^2 \times i = -1 \times i = -i, \qquad i^4 = i^2 \times i^2 = (-1) \times (-1) = 1$$

Exercise 1: Simplify i^8 and i^{11}.

$$i^8 =$$

$$i^{11} =$$

We treat i like other numbers in that we can multiply it by numbers, we can add it to other numbers, etc. The difference is that many of these quantities cannot be simplified to a pure real number.

For example, $3i$ just means 3 times i, but we cannot rewrite this product in a simpler form, because it is not a real number. The quantity $5+3i$ also cannot be simplified to a real number.

However, $(-i)^2$ can be simplified.

$$(-i)^2 = (-1\times i)^2 = (-1)^2\times i^2 = 1\times(-1) = -1.$$

Because i^2 and $(-i)^2$ are both equal to -1, they are both solutions for Equation 2 above.

3 The Complex Plane

Definition: A complex number is one of the form $a+bi$, where a and b are real numbers. a is called the real part of the complex number, and b is called the imaginary part.

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal, i.e., $a+bi = c+di$ if and only if $a=c$ and $b=d$.

Example 2. Here are some complex numbers:

$$2-5i, \qquad 6+4i, \qquad 0+2i=2i, \qquad 4+0i=4.$$

The last example above illustrates the fact that every real number is a complex number (with imaginary part 0).

It is often useful to think of real numbers as points on a number line. For example, you can define the order relation $c<d$, where c and d are real numbers, by saying that it means c is to the left of d on the number line.

We can visualize complex numbers by associating them with points in the two-dimensional plane, which we call the complex plane $\mathbb{C}$. We do this by letting the number $a+bi$ correspond to the point (a,b), where a is the coordinate of the complex number along the *real axis* (which we usually call the x-axis) and b is the coordinate of the complex number along the *imaginary axis* (which we usually call the y-axis).

4 Complex Arithmetic

When a number system is extended, the arithmetic operations must be defined for the new numbers and the important properties of the operations should still hold for the old numbers. For example, addition of whole numbers is commutative. This means that we can change the order in which two whole numbers are added and the sum is the same: $3+5=8$ and $5+3=8$.

We need to define the four arithmetic operations on complex numbers.

4.1 Addition and Subtraction

To add or subtract two complex numbers, you add or subtract the real parts and the imaginary parts.

$$(a+bi)+(c+di) = (a+c)+(b+d)i$$

$$(a+bi)-(c+di) = (a-c)+(b-d)i$$

Example 3.

$$(3-5i)+(6+7i) = (3+6)+(-5+7)i = 9+2i$$

$$(3-5i)-(6+7i) = (3-6)+(-5-7)i = -3-12i$$

Note: These operations are the same as combining similar terms in expressions that have a variable. For example, if we were to simplify the expression $(3-5x)+(6+7x)$ by combining similar terms, then the constants 3 and 6 would be combined, and the terms $-5x$ and $7x$ would be combined to yield $9+2x$.

4.2 Multiplication

The formula for multiplying two complex numbers is

$$(a+bi)*(c+di) = (ac-bd)+(ad+bc)i.$$

You do not have to memorize this formula, because you can arrive at the same result by treating the complex numbers like expressions with a variable, multiply them as usual, then simplify. The only difference is that powers of i do simplify, while powers of x do not.

Example 4.

$$\begin{aligned}(2+3i)(4+7i) &= 2\times 4+2\times 7i+4\times 3i+3\times 7\times i2 \\ &= 8+14i+12i+21\times(-1) \\ &= (8-21)+(14+12)i \\ &= -13+26i\end{aligned}$$

Notice that in the second line of the example, the i^2 has been replaced by -1.

Using the formula for multiplication, we would have gone directly to the third line.

Exercise 2: Perform the following operations.

(a) $(-3+4i)+(2-5i) =$

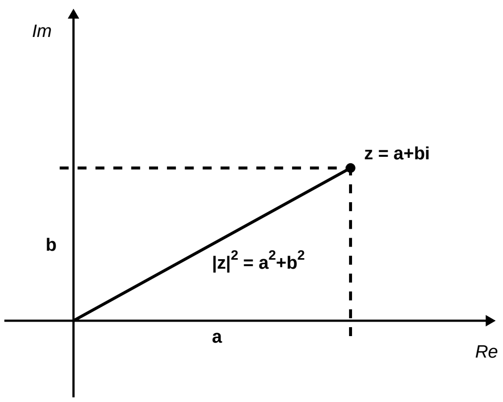
(b) $3i-(2-4i) =$

(c) $(2-7i)(3+4i) =$

(d) $(1+i)(2-3i) =$

5 Absolute value

Just as the absolute value of a real number measures the distance of that number from zero along the real number line, the absolute value of a *complex number* measures the distance of that number from zero in the *complex plane*. Remember that in the complex plane, the complex number $z = a+bi$ corresponds to the point (a,b).

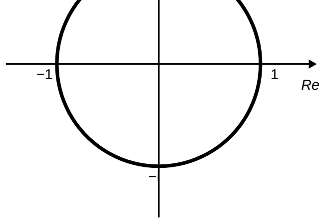


How will we find the distance of the complex number $z = a+bi$ from the origin? Observe that the length of the line from the origin to z is simply the hypoteneuse of a right triangle, with one side of length a and the other side of length b. It's simple, then, to use the Pythagorean theorem to find the length of the hypoteneuse, which is $|z|$:

$$|z|^2 = a^2+b^2$$

$$|z| = \sqrt{a^2+b^2}$$

5.1 The unit circle

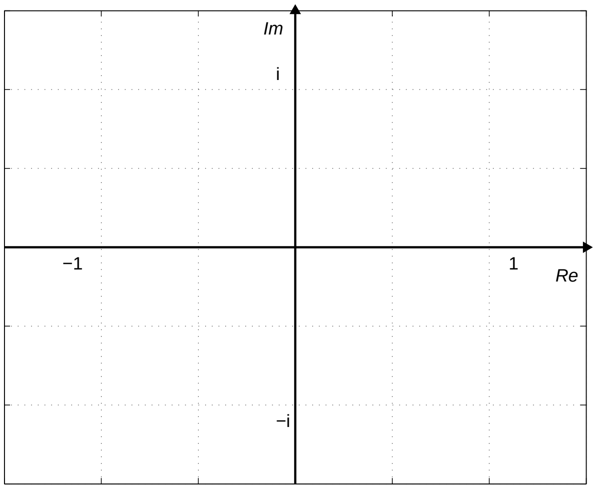


Some complex numbers have absolute value 1. Of course, 1 is the absolute value of both 1 and -1, but it's also the absolute value of both i and $-i$ since they're both one unit away from 0 on the imaginary axis. The unit circle is the circle of radius 1 centered at 0. It includes all complex numbers of absolute value 1, so it has the equation $|z| = 1$.

6 Some more practice!

Exercise 3:

(a) On the axes below, draw and label the following three points in the complex plane: $z = \frac{1}{2}i, 1-i, -1$.



(b) Calculate $|z|$ for each of the three points in (a).

Exercise 4: Every complex number $z = x + yi$ has a *complex conjugate*, the complex number $\bar{z} = x - yi$.

(a) What is $|z|$? What is $|\bar{z}|$?

(b) Compute $z \times \bar{z}$. Is your answer real, imaginary or complex?

(c) How are $|z|$ and $z \times \bar{z}$ related?

Exercise 5: If $z = x + yi$, for what values of x and y is $|z^2|$ greater than $|z|$? How do you interpret your answer geometrically, in the complex plane?