Arabesque Groups – Where Art and Mathematics Meet

Jawad Abuhlail, KFUPM (KSA)

abuhlail@kfupm.edu.sa

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Arabesque Groups

Where Art and Mathematics Meet

Dr. Jawad Younes Abuhlail

King Fahd University of Petroleum & Minerals

Segment I

In the name of God, the Merciful.

Dear viewers,

May Peace, Mercy and Blessing of God Almighty be upon you.

At the beginning, I would like to introduce myself. I am Dr. Jawad Younes Abuhlail, from the Department of Mathematics and Statistics, at King Fahd University of Petroleum & Minerals, Dhahran, Kingdom of Saudi Arabia. My main specialty is Algebra and its applications in real life.

In this module, we try to get some knowledge about the so called **wallpaper groups** or **Arabesque groups**. Some of these groups exist in many mosques and temples all over the world. However, it is worth mentioning that all of these groups exist in one place, namely the palest Alhambra in Andalusia (Granada).

Let's begin with the following definition.

- We define an **isometry** as a bijective map (*i.e.* 1-1 and onto) f from the Euclidean plane $\mathbb{R}^2$ to the Euclidean plane $\mathbb{R}^2$ which preserves distances, that is the distance between the points P and Q is equal to distance between the image of P and the image of Q for all points P and Q in the Euclidean plane.

We denote the set of these isometries by E.

If W is a subset of the Euclidean plane $\mathbb{R}^2$, then we define the set of symmetries of W as the set $Sym(W)$ of isometries in E such that the image of W is equal to W.

Notice that $E = Sym(\mathbb{R}^2)$, the set of all symmetries of the Euclidean plane, which is a group as we will show later, and that the set $Sym(W) \subseteq Sym(\mathbb{R}^2)$ of all symmetries of W is a subgroup for every subset $W \subseteq E$.

What about practicing the first activity now? What are the symmetries of the circle?

Activity 1: What are the symmetries of the circle?

Segment II

Welcome again.

I hope you were successful in obtaining the symmetries of the circle. Let's now go back to the following question: What are the symmetries of the plane?

As you notice here, if we take this rectangle and draw a line in the middle of this arc (this way), then we recognize symmetry between the left and the right parts of the rectangle so that there is a mirror reflection, *i.e.* as if we had inserted a plane mirror at the middle of this arc. Also, if we pull this rectangle in this direction, then we see that it coincides with the other rectangle. So, in this wall there is another symmetry which is translation. So, we have two types of symmetries in the plane, namely translation and the reflection (or the mirror reflection).

There is another symmetry which is rotation (counterclockwise) through an angle between 0° and 360° as shown in the accompanying animation. So, these are the three basic types of symmetries in the plane: translation, rotation through a specific angle and reflection (or the so called mirror reflection).

Notice that the composition of any two symmetries is also a new symmetry. There is a special and distinguished type of symmetries, called the glide reflection, and it is a composition of a mirror reflection followed by a translation along a vector parallel to the axis of the mirror reflection as shown in the given animation.

The question now is:

Are there symmetries that cannot be obtained as compositions of the basic symmetries, which are translations, rotations (through a specific angle) and mirror reflections?

To answer this question, let's take an arbitrary symmetry $\rho \in E$ and let's assume (without loss of generality) that $\rho(\vec{0}) = \vec{0}$. A nice property of this symmetry ρ is that it preserves not only distances but also angles. Since ρ preserve distances, the image of any circle C centered at the origin, and with radius r, is equal to the same circle C. We fix now the point $A = (r, 0)$ on this circle, where r is the radius of this circle. Notice that any point B on this circle is determined by the angle which OB makes with OA. Let this angle be β.

Let $\rho(A) = A'$ where the angle that OA' makes with OA is α and the angle which OB makes with OA is β where the image of B under this symmetry in B'.

Since ρ preserves angles, we have only two cases:

Case 1: $\widehat{AOB'} = \alpha + \beta$.

In this case, ρ represents rotation of an angle α counterclockwise.

Case 2: $\widehat{AOB'} = \alpha - \beta$.

In this case, ρ represents a reflection in the diameter CD, where the angle $\widehat{AOC} = \frac{\alpha}{2}$.

So, there is an infinite number of symmetries of the circle which are of two main types:

The first type is rotation (counterclockwise) through some angle α. The second type is reflection about one of the diameters of the circle.

Denote the reflection about the *x*-axis by the symbol F and let L be the reflection about the diameter CD, where the angle $\widehat{AOC} = \gamma = \frac{\alpha}{2}$. As you can see (from the animation), L is equal to the composition of a reflection about the *x*-axis followed by a rotation through an angle α (counterclockwise).

So, we can obtain any reflection about a diameter of the circle as a composition of the reflection about the *x*-axis and a suitable rotation. From that we conclude that any symmetry of the circle is a rotation (counterclockwise) through a specific angle or a reflection about the *x*-axis or a composition of the reflection about the *x*-axis with a suitable rotation. As for symmetries in the plane, we add to these translations alones a specific vector.

As you see, this is the so called *Shmagh* or *Koufiyya*, which is head cover which is used by some men in the Arab. What about finding ally symmetries in the shown part of the Koufiyya?

Activity 2: Find all symmetries in the displayed part of the *Shmagh*.

Segment III

Welcome again.

Can you imagine a relation between matrices, which are important tools in Algebra and Arabesque groups which are mainly related to Art. In fact, there is a very close relation. We will use matrices to prove that the set of symmetries on an Arabesque wall is a group.

We represent translation through a vector $\vec{v} = \binom{a}{b}$ by the map $T_{\vec{v}}$. As you see in the displayed animation of a picture for a wall in Alhambra, there are translations in two directions: horizontal and vertical.

Rotation (counterclockwise) through an angle α is represented by the matrix A_α. The Arabesque group of the displayed picture for a wall in Alhambra contains rotation through an angle 90° counterclockwise and is represented by A_{90}.

Denote the reflection about the x-axis by F. If P is the point (x, y), then $F(P) = (x, -y)$. Since $\binom{x}{-y}$ is the product of the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ with the vector $\binom{x}{y}$, the reflection about the line that makes an angle γ with the positive x-axis, *i.e.* a rotation through an angle 2γ following a reflection in the x-axis as we clarified earlier, is represented by the matrix B_γ.

For example, in the Arabesque group of the picture of a wall in Alhambra, there is a reflection about the line that makes 45° with the positive *x*-axis and it can be represented by the matrix B_{45}. Notice that all the matrices used in representing the symmetries belong to the group of special orthogonal matrices of order 2 which consists of the set of all square matrices of order 2 with the property that the product of each matrix with its transpose (from either side) is the identity matrix. For example, the matrix A_{90} is a special orthogonal matrix (as the displayed calculations show).

So, we can represent *E* as the given set of ordered pair and we can define on it a binary operation *.

What are the properties of this binary operation?

You are referred to the extra material on the website of this module which contains the calculations which show that:

1) the operation * is associative.
2) the ordered pair $(I, \vec{0})$ is a neutral element.
3) ever element $(A, \vec{v})$ in E has an inverse in E given by the ordered pair $(A^{-1}, -A^{-1}\vec{v})$.

What do we conclude?

We conclude that (*E*,*) is a group.

Let's move now to the third activity: draw an equilateral triangle, then find all symmetries of this equilateral triangle. Do the set of such symmetries from a subgroup of the groups of symmetries of the plane *E*?

Try to answer these questions. We will come back to you later to answer these questions.

Activity 3: What are the set of symmetries of the equilateral triangle? Does it form a subgroup of the group E of symmetries of the plane?

Segment IV

The set of symmetries for any regular polygon with n edges forms a subgroup of the group of symmetries of the plane consisting of $2n$ elements.

These symmetries are of two types: rotations and reflections about one of the reflection axes. We provide now some details about the elements of this group.

***n* is odd:** in this case, there are n axes of reflection, and these are exactly the ones connecting each vertex with the midpoint of the side opposing it (as in the case of the equilateral triangle).

***n* is even:** in this case, there are also n axes of reflection: half of them are connecting the opposite vertices, while the other half connect the opposite edges (as in the case of the square).

We will provide now a detailed explanation of the examples of the equilateral triangle and the square.

Let's begin with the case of the equilateral triangle. The symmetry group of this equilateral triangle consists of two basic types:

- The first type consists of three rotations (clockwise about the center): through angles of 120°, 240° and 360° which is equivalent to rotations through 0°;
- The second type consists of three mirror reflections about one of the reflection axes which are the lines connecting each vertex and the opposite line as shown.

So, there is a total 6 elements in the group of symmetries of the equilateral triangle.

As for the square, there are 8 symmetries:

Four rotations (clockwise about the center) through angles of 90°, 180°, 270° and 360°, which is equivalent to rotation through 0°.

As for reflections, we have four reflection axes: can you find them? The first connects the midpoint of this side with the midpoint of the opposite side. The second connects the midpoint of this side with the midpoint of the opposite side. As for the third, it connects this vertex with the opposite vertex. The fourth connects this vertex with the opposite one.

Four axes of reflections for the square. So, there is a total of 8 reflection axes for the square.

In general, the set of symmetries for a regular polygon with *n*-sides is a group consisting of 2*n* elements, half of them are rotations and the other half are reflections thought one of the reflection axes (there are *n* of them).

You might wonder now, why do we concentrate on equilateral triangles and squares.

Well, can you find other regular polygons which can be put side by side to cover a whole wall without leaving any gaps and without having any overlap between them? Try to use now regular polygons (*e.g.* regular pentagons, regular hexagons, regular heptagons) to do this. You may experiment this with your colleagues in class and we will come back to you soon.

Activity 4: What are the regular polygons, identical copies of which can be used to cover the Euclidean plane without leaving any gaps and without having any overlap between them?

Segment V

Welcome again.

Now, it is legitimate that you ask: what is an Arabesque group?

Definition: An Arabesque group is a subgroup $G \subseteq E$ containing translations in two different directions (along two non-zero non-parallel vectors, which generate a lattice $L(G)$ consisting of all vectors which constitute part of the elements in G).

In this case, the subset of all translations in G forms a subgroup of G which we denote by $T(G)$.

What comes to mind now is the following question:

How can one classify the Arabesque groups?

Notice that the subgroup $T(G)$ which consists of all translations in the Arabesque group G plays an important role in classifying this group, since it determines the shape of the original piece, identical copies of which are used to cover the wall.

Definition: Let $G, G' \subseteq E$ be Arabesque groups. We say that G and G' are equivalent ($G \approx G^{'}$) iff there is an isomorphism of groups $f: G \to G^{'}$ which satisfies $f(T(G) = T(G^{'})$.

Let's ask ourselves now:

What are the possible shapes for a lattice of the Arabesque groups?

We have the following possible shapes of the basic building piece:

Parallelogram, which might be:

Slant

Rhombus

Rectangle

Square

There is also a special distinguished shape: a rhombus, with one of its vertices the center of a rectangle; this shape is called a centered rectangle.

Are there other shapes for the basic building piece in an Arabesque group?

The following fact is very important in classifying Arabesque groups.

Fact: Let G be an Arabesque group. If G has a non-trivial rotation $R_{\frac{360}{n}}$, then $n \in \{2, 3, 4, 6\}$.

This means that the smallest non-trivial (non-zero) angle of rotation is either 180°, 120°,90° or 60°.

Proof: Let $n \geq 2$. Let P be one of the centers of rotation (there is infinite number of them) and let Q be another center of the reflection $R_{\frac{360}{n}}$, so that the length of the line segment PQ is as small as possible.

Let P' be the image of P in the rotation whose center is Q and let Q' be the image of Q in the rotation whose center is P'. Notice that length of PQ is equal to the length of QP' and is equal to the length of $P'Q'$.

First Case: $Q^{'} = P$.

In this case we get an equilateral triangle and $n = 6$. The basic building piece in the lattice in this case is a rhombus with angles 60° and 120°. One can use the edges of these triangles to build hexagons.

Second Case: $Q^{'} \neq P$. Notice that the distance between P and $Q^{'}$ is larger than or equal than the distance between P and Q (*i.e.* $d(P, Q) \leq d(P, Q^{'})$).

- $n > 6$: in this case, the distance between P and P' is less than the distance between P and Q as clear from the accompanying figure, which is a contradiction.

- $n = 5$: In this case, the angle of rotation is 72°. In this case, the distance between P and Q' is less than the distance between P and Q, a contradiction since the distance between P and Q is (by assumption) as small as possible.

- $n = 4$: In this case, the angle of rotation is 90°. In this case the basic building piece is a square.

- $n = 3$: In this case, the angle of rotation is 120°. The basic building piece in this case is a rhombus with angles 60° and 120° as shown in the accompanying figure.

- $n = 2$: In this case, the angle of rotation is 180°. The basic building piece in this case is a general parallelogram.

What about a new activity?

Activity 5: Try to find the basic building piece in each of the following pictures.

Segment VI

Welcome to the last segment of this module. It is obvious that there is an infinite number of Arabesque groups in the world. But some might ask: what is the number of non-equivalent Arabesque groups? The Russian mathematician Fedorov proved in 1891 (AD) that the number of non-equivalent Arabesque groups is 17.

Independently, the Mathematicians Polya and Niggli recovered this result in 1924. Proving this fact is beyond the scope of this module. However, we will present an example that explains how to classify Arabesque groups and how to recognize the type of that group.

Let’s consider the following picture for a wall in Alhambra. To recognize the Arabesque group in this picture, we ask ourselves few questions:

1) What is the basic building piece in this picture?

   It is obvious that the basic building piece is a rhombus with angles 60° and 120°.

2) Are there translations in two different directions?
   We notice that we have translations in two different directions and so we have an Arabesque group.

3) What is the smallest angle of a non-trivial (non-zero) rotation?

   We notice that there are rotations through an angle of 60° as shown in the animation.

   Referring to the manual on classifying Arabesque groups on the website of this module, we find that there are two possibilities: this Arabesque group is either **P611** or **P6mm**.

   What is the difference between the two groups?

   The group P611 does not contain any mirror reflections, while the group P6mm contains mirror reflections.

   Are there mirror reflections in this picture? Can you find some of them, if any?

   Well done!!

   There are mirror reflections and so the Arabesque group is P6mm.

   Some might wonder, what do these symbols P6mm mean?

- The letter **P** means that the basic building piece is primitive and not a centered rectangle.
- The number **6** means that the smallest angle non-trivial rotation is $\frac{360^\circ}{6} = 60^\circ$.
- The fist letter **m** means that there is a perpendicular mirror reflection which is shown in light blue.
- The second letter **m** means that there is a mirror reflection that makes an angle 60° with the positive *x*-axis (which is the left edge of the rhombus pointing downwards). The mirror reflection is shown in blue in the accompanying figure.

We arrived now at the final activity of this module. Let's try to end with a painting of famous Dutch painter Escher who used much symmetry in his famous paintings.

Activity 6: Try to classify the Arabesque group in this paining of Escher.

Finally, I hope you have enjoyed this module and that it was useful for you.

May Peace, Mercy and Blessing of God Almighty be upon you.

Teachers’ Guide

The main goal of this module is to relate Mathematics and Groups (in particular) with real life and art. The class activities accompanying this module are the following:

Activity 1: Trying to find the symmetries of the circle.

Activity 2: Trying to find the symmetries in the *Shmagh* or *Koufiyya* (which a head cover used by men in the Arabic world especially in hot weather).

Activity 3: Trying to find the symmetries of the equilateral triangle.

Activity 4: Trying to find the regular polygons that can be used to cover a whole wall having no overlap between them and leaving no gaps.

Activity 5: Trying to find the basic building piece in a picture for a wall in Alhambra, Andalucía (and other pictures).

Activity 6: Trying to classify the Arabesque group in one of the paintings of the famous Dutch painter Escher.

Finally, I hope you have enjoyed this module and that it will be useful for your students.

May Peace, Mercy and Blessing of God Almighty be upon you.