Part I:

Peace, mercy and blessings of God, I am Amjad Al-Noor, I would like to present this lesson for you which has been prepared by Fatma Al-Qahtani about the miraculous geometric design of the hexagonal wax cells.

Humans have been learning a lot of things by exploring the world around them. Through this attempt to understand the great designs around humans, plane designs were made similar to some birds, and the helicopter eres designed similarly to one of the insects. Today we will explore the secrets of the beehive. When we look closely inside the beehive, we find that larvae and honey are collected in small hexagonal wax cells that reflect a magnificent engineering design. This makes us wonder: How have the bees made such a design? How did the bees know that the hexagon is the best design to host the larvae? How did the bee, which lives for only six weeks, find out the accuratemeasurements and calculations? How were they able to set the standards that are not easy even for humans?

Try to find the reasons behind selecting the hexagonal design for the wax cell and I will come back to you shortly, God willing.

Part II:

Hello again, the beehive architectural design has caught the attention of humans, attracting many engineers to design buildings and skyscrapers inspired by the wax cell design. The most important aspects that have attracted attention to the work of bees is their accurate and precise ability to build, and to work as a team; every single bee participated by putting one of the parts, in full harmony with the professional engineering standards, the thickness of its hexagonal walls is around (0.1) mm, and the deviation from the average value does not exceed (0.002) mm. In order to realize the degree of accuracy for these engineering rules in the construction of wax cells, we must take a deep mathematical look. To understand more, we will mention that bees use these hexagonal cells to store honey and also to host larvae. The cells have a shape close to a cylinder, with a circular cross section like a bee. The bee needs a circular cross section to insert its body while storing honey.

Try to line circles of equal size, in the smallest possible area of the plane. For example, twenty pieces of metal within a small frame, and look for a shape of a polygon to separate the circles with no spaces or at least a minimum of space waste, and I will get back to you soon.

Part III:

Welcome once again, you might have found that when circles are lined around each other, there should be six similar circles around each circle. If the middle points of the empty spaces between them are connected to the points where they touch each other, we get what we call a regular hexagonal polygon.

Perhaps, the hexagonal polygon - as geometrically shown - is the only polygon that combines circles with minimal loss of inter-spaces, but, how can we prove it mathematically? Let us recall first the definition of so-called "regular polygons". A regular polygon is a polygon in which all the sides are equal and whose angles are also equal in measurement. Now, let us think about the type of regular polygon that we can use repeatedly to divide an area, so that there are no spaces among the polygons (and thus there is no wasted space), taking into account that the total lengths of the sides of these polygons is as little as possible (and therefore using the least amount of construction materials)

When we want to split a large area to small areas, adjacent polygons should fuse to each other without leaving blank spaces among them, as well the sum of the interior angles of the adjacent sides at a commonvertex is equal to (360) degrees, as follows:

It is well known that the measurement of an interior angle in a regular polygon is:

$$\frac{180(n-2)}{n}$$

Where n is the number of sides. If we have N of adjacent regular polygons then the sum of interior angles of adjacent sides at a common vertexis equal to (360) degrees as we mentioned earlier, and that can be mathematically stated to get:

$$N\left[\frac{180(n-2)}{n}\right]=360$$

$$\Rightarrow N\left[180(n-2)\right]=360n$$

And by dividing both sides by $360n$, we get the following equation:

$$N\left[\frac{1}{2n}(n-2)\right]=1$$

$$\Rightarrow N=\frac{2n}{n-2}$$

What we are trying to get here is the whole number N from the number of sides and we can get the integer value only when $n=3,4,6$, while you cannot get any integer value from the values for n that is greater than 6, which means if we want to divide an area without leaving spaces, we must use the equilateral triangle or square or a regular hexagon. We cannot use a regular pentagon, because when pentagons are aligned they leave blanks and do not complete (360) degrees.

This is only a part of the design problem that faced bees, but the other problem is how can the bees choose the appropriate design from all these three geometric figures to collect the larvae with their circular cross section inside so that bees can use the given space efficiently with a minimum loss of unused space? Which of these three designs is better?How can we make sure it is really the best?

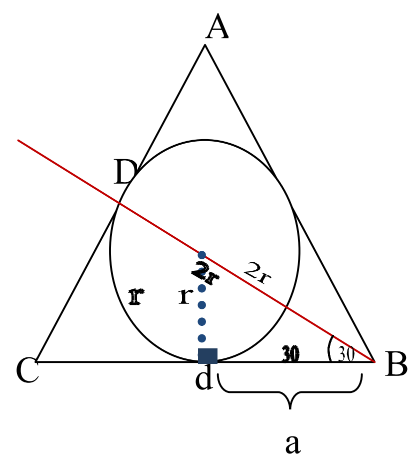
Think about this question for a few minutes and I will get back to you shortly.

Part IV:

Welcome back

We will calculate the area of each geometric figure (equilateral triangle, square and regular hexagon) containing the same circle, that is, if we have a circle of radius r, we will find the area of the equilateral triangle that contains the circle, we want to calculate the area ofthe triangle (whose sides are tangentialto the circle) in terms of r, as well as for the square and the regular hexagon, and then compare them to find which one is the best design for a given area.

First, the equilateral triangle:



• Suppose that the length of line segment (side): $BC = 2a$

• BD is the height that divides AC into two equal halves

And also divides the angle B

• The radius of the circle is r

• C is the center

Note that - in this figure - we have a right triangle,

It is known that in the triangle whose angle is 30:

The side oppositeto the angle 30 = half the hypotenuse

Thus, the hypotenuse = 2r

According toPythagorean Theorem: $(cB)^2 = (cd)^2 + (dB)^2$

Thus, $(2r)^2 = r^2 + a^2$

$$\Rightarrow 4r^2 = r^2 + a^2$$
$$\Rightarrow a^2 = 4r^2 - r^2$$
$$\Rightarrow a^2 = 3r^2$$
$$\Rightarrow a = \sqrt{3}r$$

Since the length of the line segment (side) $BC = 2a$

$$\Rightarrow BC = 2\sqrt{3}r$$

It is known that the area of an equilateral triangle is:)Length of one side)$^2 \frac{\sqrt{3}}{4} \times$

$\therefore Area\ of\ \Delta ABC = (BC)^2 \frac{\sqrt{3}}{4}$

$$Area = (2\sqrt{3}r)^2 (\frac{\sqrt{3}}{4})$$

$$Area = (4 \times 3 \times r^2)(\frac{\sqrt{3}}{4})$$

$$Area = 3\sqrt{3}r^2$$

Second, the Square:

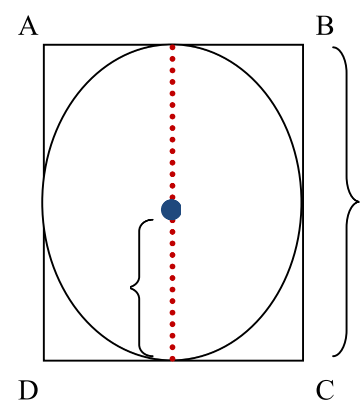
See the figure to the right and note that the radius of the circle = r

Thus, the side length of the square= Diameterof the circle = 2r

It is well known that the area of square = (length of one side) ^{2}

$$\therefore Area = (2r)^2$$

$$\Rightarrow Area = 4r^2$$



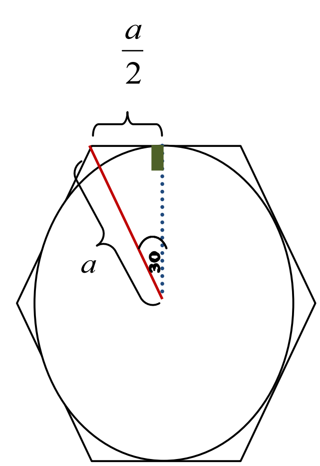
Third, the Hexagon:

Note - from the figure at the right- that we have a right triangle,

It is known that in the triangle whose angle is 30:

The side opposite to the angle 30 = half the hypotenuse = $\frac{a}{2}$

According to Pythagorean Theorem, we find that:



$$a^2 = r^2 + \frac{a^2}{4}$$
$$\Rightarrow a^2 - \frac{a^2}{4} = r^2$$
$$\Rightarrow \frac{3}{4}a^2 = r^2$$
$$\Rightarrow a^2 = \frac{4}{3}r^2$$
$$\Rightarrow a = \frac{2}{\sqrt{3}}r$$

Thus, $a = \frac{2\sqrt{3}}{3}r$

Since the area of a hexagon is: $\frac{3\sqrt{3}}{2}a^2$

$$\therefore Area = \frac{3\sqrt{3}}{2}(2\frac{\sqrt{3}}{3}r)^2$$
$$\Rightarrow Area = \frac{3\sqrt{3}}{2}(\frac{4\times 3}{9})r^2$$
$$\Rightarrow Area = 2\sqrt{3}r^2$$

By comparing the area of the previous figures, we find that the hexagon is the ideal polygon to contain circles with minimal loss of inter-spaces. But, what is the most appropriate design in terms of minimum consumption of building materials? Note that bees use wax to build the sides of each polygon within the beehive.

Discuss this question for a few minutes and I will get back to you shortly.

Part V:

Hello, again.

We will calculate the perimeter for each geometric figure (equilateral triangle, square and regular hexagon) containing the same circle, that is, if we have a circle of radius r, we will find the perimeter of the equilateral triangle that contains the circle. We want to calculate the perimeter of the triangle (whose sides are tangential to the circle) in terms of r, as well as for the square and the hexagon, and then compare them to find which one is the best design that uses the minimum amount of building materials.

It is known that the perimeter of regular polygons equals the sum of the lengths of their sides, i.e., na

Where: *n* represents the number of sides, *a* represents the length of one the sides

Thus, the perimeter of equilateral triangle = $3(2\sqrt{3}r)$

= $6\sqrt{3}r$ =

= 10.932 r

The perimeter ofsquare =4(2r)

=8 r

The perimeter of hexagon = $6(\frac{2\sqrt{3}}{3}r)$

= $4\sqrt{3}r$ =

= 6.928 r

Thus, we have found that hexagon is the best design in terms of space and less needed building materials, and therefore chosen by bees to savespaces and reduce the production of wax since the production of a singlekilogram requires the same effort from bees to produce 10 kilograms of honey!

Part VI: (6 minutes)

We'll show you now a two-dimensional description of honeycombs, but in fact most people are not aware that the three-dimensional wax cell structure is the most fascinating.

Technically, the basic unit of a honeycomb is a special type of hexagonal prism with a hexagonal top base and a convex lower base and consists of three rhombuses. When many of these basic units are pasted together, they form beehives with their well known shape.

The French astronomer Giacomo Filippo Maraldi was the first scientist interested in measuring the angle of the basic unit of the beehive structure in 1712, and concluded that the angles between the rhomboid bases and the hexagonal prism are always consistent and exactly equal to $70^\circ 32'$ and pointed out that bees have used this angle to keep the simplicity of construction.

But the French biologist, René Antoine Ferchault de Réaumur had another opinion about the selection to that angle and supposed that reason was for saving the amount of wax consumed to build cells.

To prove what he assumed, he wrote to many mathematicians to ask about the best angle to link rhomboid bases and hexagon prism that uses the minimum amount of wax. The only one who answered his question was the Swiss mathematician Johann Samuel König and the result was $70^\circ 34'$

Which differs with the result of Maraldi by 2' "two minutes ".

In 1743, the Scottish mathematician, Colin Maclaurin introduced a solution by a geometric method that found out the angle between the three rhomboid bases and the hexagon prism is $70^\circ 32'$.

This result is similar to the result referred to by Maraldi.

Now, that we know the angle between the three rhomboid bases and the hexagon prism is $70^\circ 32'$, we will try to prove that the beehives provide the greatest amount of honey when the top base is hexagonal and the bottom is convex and consists of three rhomboid bases at an angle of inclination equal to $70^\circ 32'$.

Suppose that the hexagonal prism has two flat bases as shown.

Let us assume that AB = a, A'' A = b, then we can calculate the surface area S_0 and volume as follows:

$$S_\circ = 6ab + \frac{3\sqrt{3}}{2}a^2, V = \frac{3\sqrt{3}}{2}a^2 b$$

Let the distance between the two points B'B'' = x, now let's cut off the slide connecting the two points A'' C'' passing through the point B' and then we re-arrange these pyramidal slides to the top of prism.

If this procedure has been repeated for the other two sides of the prism, the prism will end up with three rhomboid geometric bases as shown below.

Note that the volume has not changed, but the surface area of this prism differed from the previous one S0, and therefore the new surface area is given as follows:

$$S_1 = 6ab - 6(\frac{1}{2}ax) + 6(\text{Area of } \Delta A''C''B') \quad ....*$$

From Al-Kashi theorem, named after the mathematician Ghayath Al-din Kashi, we find the length of the third side in a triangle when one of the angles and two sides are known by using the following formula:

$$\sqrt{a^2 + b^2 - 2ab\cos\theta}$$

Then: $A''C'' = \sqrt{a^2 + a^2 - 2a^2 \cos 120} = \sqrt{3a^2} = \sqrt{3}a$

$$B'C'' = \sqrt{x^2 + a^2}$$

Height of $\Delta A''C''B' = \sqrt{(B'C'')^2 - (\frac{1}{2}A''C'')^2}$

$$= \sqrt{x^2 + a^2 - \frac{3}{4}a^2} = \sqrt{x^2 + \frac{a^2}{4}}$$

Since the area of the triangle is equal to: $\frac{1}{2} \times base \times high$

Then: Area of $\Delta A''C''B' = \frac{1}{2}\sqrt{3}a\sqrt{x^2 + \frac{a^2}{4}}$

By substituting in the previous equation, we find that:

$$S_1 = 6ab - 6(\frac{1}{2}ax) + 6(\frac{1}{2}\sqrt{3}a\sqrt{x^2 + \frac{a^2}{4}})$$

$$\Rightarrow S_1 = 6ab - 3ax + 3\sqrt{3}a\sqrt{x^2 + \frac{a^2}{4}}$$

Assuming that the edge of the hexagon is constant "a" and can be given the value of 1 for example, and the height is constant "b" as well and suppose it is equal to 1, and if we give different values for x to locate the pieces, such as: 0, 0.1, 0.2, 0.3, 0.4,… and we calculated area which is then represented graphicallyas a curve, we will find that the value of the area is at the minimum and hence the least amount of wax is used when the value is between 0.3, 0.4.

To find out the exact value, we find the difference between the surface areas S_0 and S_1 as follows:

$$\Delta S(x) = S_0 - S_1$$

$$= \frac{3\sqrt{3}}{2} a^2 + 3ax - 3\sqrt{3a}\sqrt{x^2 + \frac{a^2}{4}}$$

Then, we find the first derivative for ΔS :

$$\Delta S'(x) = 0$$

$$3a - 3\sqrt{3}ax(x^2 + \frac{a^2}{4})^{-\frac{1}{2}} = 0$$

The solution for this equation is

$x = \frac{a}{2\sqrt{2}}$,since the value of a is 1, then x= 0.35355339, and by relating this value to the geometric design for beehives, we will understand why beeshave chosenthe angle to be $70^\circ 32'$.

This is not everything. The magnificent geometric designs of beehives are subject to overlap with similar designs without spaces, and opposite to each other so that the center of the cell is supported with three walls from the back, giving it great strength and as little consumption of wax as possible.

Now, do not you agree with me that bees are among the most skilled and talented engineers on this planet? It has been a great pleasure to present this lesson and I hope you have enjoyed it.

Peace, mercy and blessings of God