Salam. My name is Mohammad Zuheir Abu-Sbeih from King Fahd University of Petroleum and Minerals.

I would like to welcome you to this BLOSSOMS lesson in mathematics and I hope that you are feeling good and full of energy.

Today we have an interesting and challenging problem to discuss, which will use the skills that you acquired in school and will be different from what you learned in algebra, geometry and arithmetic. The topics we will discuss today are part of an area of mathematics called Graph Theory. This theory has many applications in electrical circuits, computer programs, and scheduling of aircraft and trains.

Let's start with a simple example related to three utilities: We have three houses and three stations: power, water and gas. We want to connect each of these houses to each of the stations. Can we do this without the lines crossing each other?

I would like for you to try this with your peers in class. See if you can connect the three stations to the three houses without the lines crossing each other.

I will see you after few minutes

Activity -1

Welcome back. Let’s see if you were able connect the three stations to the three houses without the lines crossing each other.

As you may have found, it is not possible. What if there are more than three stations and more than three houses, the situation will be more complicated and difficult.

To represent the problem mathematically, we may represent the three stations with three points in the plane and we call them (S_1, S_2, S_3) and the three houses with three points, say (H_1, H_2, H_3). What we want to do is to connect each of these stations to each of the three houses. As you know, pipes or electrical wires do not always go in straight lines .For example, we may connect this point with this, and this point with this. The resulting shape is called a “graph” and consists of a set of points called **vertices**; the three at the top represent the stations, and three below represent the houses, and the set of lines are called **edges**.

In general, we define the **graph G**as a pair consisting of two sets: a set of vertices (V) and another set of edges (E), so that each edge connects two different vertices.

Looking at this example, we have a set of vertices V= {V_1, V_2, V_3, V_4, V_5, V_6} and also a set of edges E = {e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8}.

So this graph consists of 2 sets of vertices and edges as you see in the figure.

If an edge$_1$e connects two vertices $_1$V and $_6$V for example, or $_2$econnects the two vertices$_2$V and$_5$Vwe write $_1$V$_6$V= $_1$e. As well as $_5$V$_2$V= $_2$e.

Now, I want you to draw a graph that contains five vertices and ten edges so that no more than one edge links between2 different vertices. Compare the graph you've got with your colleagues in the group to see if you get the same graph or not.

I'll see you after a while.

Activity – 2

Welcome back. I expect that you got the same graph as you can see on the blackboard. This graph will be called "the **complete graph** on 5 vertices" and will be denoted by k_5. This graph consists of five vertices and any two different vertices are connected by one edge.

There are two important families of graphs. The first family is the **complete graph** K_n, which has *n* vertices, and any two vertices are connected by an edge. The other family is called the **complete bipartite graph**. The complete bipartite graph consists of a set of vertices divided into two subsets *A* and *B*, and a set of edges connecting each vertex in *A* to each vertex in *B*.

In general, if we have *m* vertices in the first vertex set *A* and *m* vertices in the other vertex set *B*, the complete bilateral graph will be denoted by$_{,m}K_n$.

There is another interesting property of graphs, which is called connectedness. A graph *G* is said to be **connected** if any two vertices in G are connected by an edge, or a set of consecutive edges. The set of different consecutive edges is usually called "**path**". Let's look at the graph G on the board; we note that between any two vertices, either there is an edge connecting them such as$_6$econnects$_1$V to $_2$V; or there is a path connecting them such as the path {$_8$e, $_3$e} connects$_1$Vto$_4$V. We may have more than one path connecting a pair of vertices. Other paths connecting $_1$Vto$_4$Vare {$_4$e, $_5$e, $_1$e} and {$_4$e, $_2$e, $_6$e}. While in this figure: this graph doesn't have any edge or path connecting the vertex U to V; so they are separated. This graph is **not connected** while this one is.

In this lesson we'll consider connected graphs only.

Now try to draw each of the complete graphs

$_5$K , $_4$K , $_3$K , $_2$K , $_1$ K and the complete bipartite graphs $_{3,3}$K , $_{2,2}$K , $_{1,2}$K , $_{1,1}$Kon paper.

Which one of these graphs can be drawn in the plane or on a paper without its edges intersecting each other?

Activity - 3

Again welcome back. Let us see if we got the same result. You may noticed that the two graphs $_{3,3}K$ and$_{5}K$ cannot be drawn in the plane without the edges crossing each other. This leads to the following definition:

A **planar graph** is a graph that can be drawn in the plane without its edges crossing each other. A non-planar graph is a graph that cannot be drawn in the plane without the edges crossing each other.

We saw in the previous activity that $_{3,3}K$ and$_{5}K$ are non planar graphs and cannot be drawn on a paper without the edges crossing each other. Also note that any graph containing one of these two graphs as a part of it will also be non planar. For example $_{3,4}K$ contains$_{3,3}K$and so it is non planar graph. Also the graphs $_{7}K$, $_{6}K$, $_{8}K$; or any complete graph with more than 5 vertices will be non planar graph because they contain $_{5}K$ as part of each one of them. Let's look at this graph. We may draw the graph on paper without the edges crossing each other, and so it is planar graph.

By going back to the problem of three houses and three stations which can be represented by the complete bilateral graph $_{3,3}K$, we can see that this problem cannot be solved because $_{3,3}K$is non planar graph.

Let us define another type of graphs, the plane graph.

Definition: A **plane graph** is a planar graph drawn in the plane without the edges crossing each other.

For example, this graph is a planar graph because we can draw it in the plane without the edges crossing each other. So it is planar but not plane graph. If we take this edge, which crosses the other edge, and draw it outside, we get a plane graph.

This graph is a plane graph while this graph is planar but not plane graph. Here the edges do not cross each other, while in this graph they intersect at this point.

Question: what will happen if we cut the paper along the edges of a plane graph?

In the given paper, I want you to cut the paper along the edges of the graph and find a relation between the number of vertices, edges, and the pieces of the paper.

I'll see you in a while.

Activity – 4

Welcome back. I hope that you found a relation between the number of vertices, edges and pieces. We notice first that when we cut the paper along the edges of a plane graph, we get a set of pieces with finite area and bounded by the edges of the graph, and one piece, the outside one. We call these areas the faces and denote them by F.

Next, we see that in the given graph the number of vertices is 4, number of edges is 6 and the number of faces is 4. Therefore, the number of vertices minus the number of edges plus the number of faces is 2.

Let us take another example. In this plane graph, what is the number of vertices, edges and faces?

The vertices are 1,2,…,8 ; i.e. we have 8=|V|, and there are 1,2,...,12 edges; i.e. 12=|E| and what is the number of faces? 1,2,3,4,5 and one external 6.

So 6=|F| .

Thus | F |+|E|-| V |=2

In general this equation, which relates the number of vertices, edges and faces, is called **Euler's Formula** and we can prove it by induction on the number of edges in plane graphs. I will leave it to you to prove later.

Now we have Euler's Formula which states: if a plane graph has v vertices, e edges and f faces then $\boxed{v - e + f = 2}$.

We can apply Euler's Formula for planar graphs after we redraw them in the plane without the edges crossing each other.

Now, what about graphs drawn on a sphere or on a balloon?

For example, can we apply Euler's Formula for this graph drawn on the balloon without the edges intersecting each other?

As you can see there is a set of vertices, a set of non-intersecting edges and a set of faces.

Also on this ball, if we consider these to be vertices and these are the edges and faces, can we apply Euler's Formula for these graphs?

What about graphs drawn on a tube? If we draw a graph as you can see without the edges crossing each other, can we apply Euler's Formula here?

Question:

1. Can we apply Euler's Formula on graphs draw on the sphere?
2. Can we apply Euler's Formula on graphs draw on the tube?
3. What is the relation between the number of vertices, edges and faces in each case?

Activity – 5

Welcome back and I hope that you found the relation between the number of vertices, edges and faces for graphs drawn on spheres and on tubes.

On the sphere we have the same relationship as the plane. The number of vertices minus the number of edges plus the number of faces is 2.i.e.; $\boxed{v - e + f = 2}$.

While on the tube, this number is zero. That is $\boxed{v - e + f = 0}$.

This is a real difference between graphs drawn on spheres and those drawn on tubes.

Let us move to another interesting subject, map coloring. You learned in geography how to color maps. If we have a map with different countries, we want to color the map so that any two adjacent countries (share common border) receive two different colors.

Question: What is the minimum number of colors you need in order to color the different regions of the kingdom so that any two regions with a common border receive different colors?

Is this minimum number sufficient for all plane graphs?

Try to find this number and I'll see you in a minute.

Activity -6

Welcome back students. I hope that you found the minimum number of colors needed. This number is 4 as you can see in the map.

Let us take another example. In this plane map, we can color the faces with 4 colors. We color the first face green, the second face next to it red, and this face blue and so on. The exterior face can be colored black.

As we saw 4 colors are sufficient to color the regions of the kingdom and any plane map.

Theorem (this theorem is called the **Four Color Theorem**): four colors are sufficient to color the faces of any plane map such that adjacent faces (share common edge) receive different colors.

We note here that this theorem is easy to state, but the proof is very hard.

How did some of the engineers use graph theory?

Let us look at the motherboard of a computer. Look how many vertices and connections there are on the front and the back of the board.

How did the scientists avoid the crossing between the different connections?

You may already know that the electrical circuits on the board are machine printed. How did the engineers avoid the crossing between the different connections?

Let us take a simpler example, the motherboard of a calculator. There are many electrical circuits that can be drawn as a graph. This is the corresponding graph to the calculator's motherboard. I want you to work with your group to make the connections on the given board without the wires crossing each other, because any contact between wires may destroy the circuit.

Activity-7

Welcome back. I hope you were able to make the connection. To avoid a connection, we make a hole in the board and make the connection on the back. With this we come to the end of this lesson, where we discussed the following:

- Properties of planar graphs
- Euler's formula which relates the number of vertices, edges and faces in the equation $|F|+|E|-|V|=2$
- The four color problem: 4 colors are sufficient to color any plane map so that adjacent faces receive different colors
- How to avoid crossing in electrical circuits.

I hope that you found graph theory both interesting and enjoyable and that this lesson has encouraged you to investigate more topics in mathematics and in the world around us.

Salam.

Teachers Guide

Dear teacher.

In this lesson we shall discuss some topics in graph theory. This lesson was designed to be different from what students learned in school specially algebra, geometry and arithmetic.

The goal of this lesson is to raise the student's level of thinking and to show them some applications of mathematics. This may motivate them to learn and enjoy mathematics.

We shall discuss the following concepts:

- Properties of planar graphs
- Euler's formula which relates the number of vertices, edges and faces in a planar graph
- The four color theorem, which says that 4 colors are sufficient to color the faces of any plane map in such a way adjacent faces receive different colors.
- How to avoid crossings in electrical circuits as an application of graph theory.

Through this lesson, students are expected to do some in class activities. At the beginning of the class you may divide students into groups of 3 to 5 students each. This is a summary of activities that students are asked to do.

Activity -1 :(2 minutes) Ask 3 students to represent the stations and another 3 to represent the houses. The rest of the class may try to link each station to each house through ropes or wires with different colors: one color for water, one for gas and one for electricity.

Activity -2 :(2 minutes)we ask each student to draw a graph with 5 vertices and 10 edges so that no more than one edge connects two different vertices. Compare the graph with his colleagues in the group to see if they got the same graph.

Activity -3 :(3 minutes) we ask the students to draw the graphs K_5, K_4, K_3, K_2, K_1 and the complete bipartite graphs $K_{3,3}$, $K_{2,2}$, $K_{1,2}$, $K_{1,1}$; discuss with the group which of these can be drawn on the paper with the edges crossing each other.

Activity -4 :(3 minutes) students will be given a sheet with a graph (b) drawn on it and we ask them to cut the paper along the edges of the graph. We ask them to see if there is a relation between the number of vertices, edges and pieces. Discuss your finding with your friends in the group.

Activity -5:(3 minutes) we ask students to see if we can apply Euler's formula on graphs drawn on the sphere. Try to find a relation between the number of vertices, edges and faces for graphs on the sphere and on the tube.

Activity -6 :(2 minutes) we distribute a map of the kingdom's regions and ask them to color the map with minimum number of colors. What is this minimum number? Is it the same for all plane graphs?

Activity -7 :(3 minutes) we distribute carton boards and wires (or threads) and ask them to connect the vertices as shown on the screen without the wires crossing each other. If they cannot do that, let them think of a way to connect the vertices without the edges crossing each other. What method will they use? Will this lead to see how scientists resolve the problem of edge crossing on a motherboard?

As extra work for advanced students, we may ask them to prove Euler's formula for plane graphs using mathematical induction on the number of edges. We may remind them of the 3 steps of math induction and ask them to prove the formula.

Thank you very much.