The Pythagorean_Theorem: Geometry's Most Elegant Theorem

[00:00:00.00]
[00:00:20.79] PROFESSOR HAUPT: Hello. My name is Sandra Haupt, and I teach math at Concord-Carlisle High School, located in Concord, Massachusetts. Concord's a town with a rich historical tradition. The first battle of the American Revolution was fought here, at the Old North Bridge, known as the shot heard 'round the world. And Concord's been home to many famous literary figures, including author and philosopher Henry David Thoreau. Today we're going to learn about the Pythagorean theorem, a math theorem with a rich historical tradition.
[00:00:54.93] It's been around for thousands of years. In China, in India, and in the Middle East, actually, before Pythagorean discovered it, 25 centuries ago. Why was this theorem important to so many great civilizations? Is it still relevant today?
[00:01:12.33] The Pythagorean theorem has captured our imagination like no other theorem. It's been proven literally hundreds of different ways, been mentioned on television shows from The Colbert Report to The SiImpsons, and even been in the movies.
[00:01:26.89] SCARECROW: Sum of the square roots of any two sides of an isosceles triangle is equal to the square root of the remaining side.
[00:01:31.79] PROFESSOR HAUPT: Well, that certainly sounds impressive. But it is the scarecrow correct? We'll see.
[00:01:37.35] We can prove the Pythagorean theorem using algebra and/or geometry. We can prove it using puzzles that require no words or even symbols to understand. We can even prove it using origami. You may already know something about the Pythagorean theorem. Looking at this illustration, you can see we've constructed three squares from the sides of a right triangle. A small square, drawn from the short side, or 'a', a medium square, drawn from this medium side, or 'b', and a large square, drawn from this longest side, or 'c'. How do the areas of the three squares compare to each other? Specifically, how do the areas of the small and the medium square compare to the area of the largest square? And how could you go about proving that?
[00:02:29.28] After you've written down your ideas, talk it over with a partner. I'll be back in a few minutes.
[00:02:45.26] Hi. This is Kylie. She's a student of mine at Concord-Carlisle High School, and she's with me today to help explain this lesson.
[00:02:53.32] KYLIE: Here's another way to compare areas. If the squares were to be covered in a layer of gold foil of equal thickness, would you rather the gold covering the small and medium squares, or the gold covering the large square? Well, it turns out they're equal. That is, the area of the small square, 'a' squared, plus the area of the medium square, 'b' squared, equals the area of the large squared, 'c' squared. 'a' squared plus ' b' squared equals 'c' squared. The sum of the squares-- not the square roots-- of the two sides of a right-- not isosceles-- triangle equals the square of the remaining side. Sorry, Scarecrow. That's the Pythagorean theorem.
[00:03:32.88] How can we prove this? There's many different ways. We could cut out the squares and weigh them. We could measure the lengths of the sides and calculate the areas of the squares. Or we could think of it like a puzzle, and rearrange the pieces to show how the areas are equal.
[00:03:48.98] PROFESSOR HAUPT: This wooden puzzle, made by a former student, illustrates how we can rearrange the pieces with an area of 'a' squared, and the piece with an area of 'b' squared, to show that they equal 'c' squared.
[00:04:02.61] I'm here at Killian Court at MIT in Cambridge, Massachusetts. Ancient cultures, like the Greeks, were familiar with the concept of right angles, of two segments being perpendicular to each other. They knew if you dropped something gravity would cause it to fall vertically, like this plumb-bob. And if you looked out in the distance, you saw the horizon. Vertical and horizontal are perpendicular to each other. That is, they form a right angle. If you make a triangle with one side vertical, 'a', and once side horizontal, 'b', you form a right angle. We call those two sides the legs. The third side joining them, the longest side, across from the right angle, is called the hypotenuse, or 'c'.
[00:04:53.76] The earliest proof of the Pythagorean theorem is attributed to Shang Gao, a mathematician astronomer during the Zhou dynasty, about 3,000 years ago. The Chinese mathematicians thought of a right

triangle in terms of a vertical piece, such as a gnomon, and its horizontal base, or shadow. The base or shadow is gou, and the height, or gnomon, is gu. The hypotenuse joining them is called yuan, or bowstring.
[00:05:24.11] There are many ways to prove that ‘a’ squared plus ‘b’ squared equals ‘c’ squared. This was an early proof also attributed to Bhaskara, an Indian mathematician and astronomer. In this diagram a right triangle has been copied four times so that its hypotenuse, ‘c’, forms the sides of a square. What is the total area? ’c’ squared. What about the square formed in the center? Well, each side of the smaller square is ‘b’ minus ‘a’.
[00:05:52.80] Working with a partner, carefully cut out the five pieces. This is called a dissection proof. You'll need to rearrange the five pieces to form a square with an area ‘a’ squared and a square with an area ‘b’ squared in order to show that ‘a’ squared plus ‘b’ squared equals ‘c’ squared. Don't be discouraged if you don't see the correct arrangement right away. This is one of the great fundamental discoveries in mathematics. After trying several different arrangements, if you're still stuck, your teacher may provide you with a hint.
[00:06:35.60] So how'd you do? The key to solving this puzzle is to arrange the two right triangles together along their hypotenuse so that they form a rectangle. Place the two rectangles perpendicular to each other, forming an L shape. Place the middle square along where they join. Now, instead of seeing the five original pieces, think of two squares. The small square has the side ‘a’. The medium square has the side ‘b’. Because no area was gained or lost, the area of the small square, ‘a’ squared, plus the area of the medium square, ‘b’ squared, equals the area of the original square, or ‘c’ squared. Thus, we have proved that the sum of the squares of the two legs of a right triangle equals the square of the hypotenuse, or ‘a’ squared plus ‘b’ squared equals ‘c’ squared.
[00:07:29.80] Let's look at the problem another way. What if we took a different right triangle and oriented them such that the medium side, ‘b’, plus the short side of ‘a’ together formed the length of the outer square. This time, let's prove the Pythagorean theorem algebraically. First calculate the total area, which is a square with sides ‘a’ plus ‘b’. Remember to use the distributive property when multiplying two binomials. This total area will equal the sum of the areas of the five individual pieces. Calculate the individual areas, combine like terms, and simplify, to prove the Pythagorean theorem algebraically.
[00:08:23.90] As your teacher explained to you, the total area is equal to the sum of the individual areas.
[00:08:32.05] Now we're going to do some calculations. The Pythagorean theorem is an equation with three variables. If we know any two of the variables, we can solve for the missing third one. So if we know that a triangle has a side of three units and a side of four units, take a moment to solve for what the hypotenuse, or ‘c’, will be.
[00:09:02.10] OK. You should've been able to calculate that ‘c’ has a length of 5 units.
[00:09:08.60] Here's a visual solution showing that 3 squared plus 4 squared equals 5 squared. Whole numbers that satisfy the Pythagorean equation, such as 3, 4, and 5, are called Pythagorean triples. Does this information have any practical uses? We'll see.
[00:09:26.04] I'm here along the beautiful Charles River in Cambridge, Massachusetts. Unlike the Charles River, every year the Nile would overflow its banks, flooding and covering everyone's property with mud. So how would the ancient Egyptian surveyors reestablish property boundaries? Using the Pythagorean theorem. If you want to measure a rectangle accurately, you have to make sure that the length and the width are perpendicular. And the way to make sure that two sides are perpendicular, to be sure they form a right angle, is to use the Pythagorean theorem. Think about the three, four, five Pythagorean triple. How could that enable ancient Egyptian surveyors, using just a rope, to construct a right angle and reestablish property boundaries after the annual Nile River flood? What strategy could you come up with to form a right angle given just a piece of string and a marker?
[00:10:34.36] Was anyone able to figure out a method? A rope with marks creating 12 equal increments could be formed into a right triangle with sides three, four, and five units. The angle between the sides with the lengths of three units and four units would be a right angle. By combining two of these triangles, this marked rope method could be used to construct the base of a pyramid that's a perfect square. In fact, a rope with 12 equal increments was an essential tool for surveying and for construction up through medieval times. And in the United States today we still measure using a ruler that's divided into 12 equal increments.
[00:11:21.49] Mathematicians have been calculating Pythagorean triples for a very, very long time. Here at Columbia University in New York City is a Babylonian clay tablet known as Plimpton 322. It's almost 4,000 years old.

[00:11:39.67] The Babylonians wrote using wedge-shaped marks known as cuneiform, using the number 60 as a base rather than 10. Probably because 60 has so many multiples, including 12. This may seem odd, but we still use 60 as a base today in calculating degrees of longitude and minutes and seconds.
[00:12:00.76] This clay tablet shows a list of Pythagorean triples. It may have been a teaching tool, like a worksheet. 15 centuries later Euclid, a Greek mathematician living in Alexandria, compiled a mathematical treatise known as Euclid's Elements. In addition to rigorous proofs of the Pythagorean theorem, Elements has the first written formula for calculating Pythagorean triples. But had the Babylonians discovered this formula earlier? What do you think?
[00:12:33.18] Now, on your worksheet, calculate the missing side of the right triangle. While you're doing that, be sure to look for patterns. And I'll be back in a few minutes.
[00:12:58.12] Now that you've finished the worksheet, did you notice any patterns? Hopefully you saw that some of the Pythagorean triples were multiples of each other. But what about the ones where the lengths were not all whole numbers, specifically the triangle where the lengths were 1 and 1? Well, what about the example where the sides of the triangle are each one unit? Well, the worksheet asks you to give an exact answer. So your answer must be the square root of 2. Now if you enter square root of 2 in your calculator you'll get a decimal answer, like 1.41421, which is close to square root of 2. But even if your answer is calculated to a million decimal places, it's still only an approximation. How is it possible to have a number that can't be calculated exactly, a length that can't be measured exactly?
[00:13:49.67] The Pythagoreans believed that integers and their simple combinations, what we call fractions, defined and gave order to the universe.
[00:14:01.87] Harmonious proportions of vibrating strings make musical notes that sound pleasing to the ear. Harmonious proportions make buildings and sculptures that look pleasing to the eye. They believed that these numbers and their ratios could describe the orbit of the planets, what Pythagoras referred to as the harmony of the spheres. So a number like square root of 2, that couldn't be calculated exactly, a number that couldn't be represented by a fraction or a ratio, was profoundly disturbing. It was vulgar. It seemed to defy the harmony of the universe. Some say that the Pythagoreans, an ancient brotherhood of mathematicians and mystics, swore an oath not to reveal the existence of this new, this irrational type of number.
[00:14:49.14] Were other cultures aware of irrational numbers? This Babylonian clay tablet, YBC 7289, is in the collection of Yale University in New Haven, Connecticut. It was created about 4,000 years ago. This tablet is a unique geometrical diagram of a square and its diagonals, which also forms a right triangle with the diagonal being the hypotenuse. Since the sides of the square are the same length, the ratio of the lengths of the legs to the hypotenuse is 1, 1, square root of 2. Marked along the hypotenuse in cuneiform is 1 plus 24/60 plus 51 over 60 squared, plus 10 over 60 cubed. Converted to our decimal system, this calculates square root of 2 accurately to the millionths place. Remember, this was done about 4,000 years ago.
[00:15:47.51] This clay tablet was like a worksheet. The teacher would draw the diagram on one side, and then on the backside the student would copy and redo the work. An equally accurate approximation can be found in another mathematical treatise known as The Shulba Sutras, which is written in Sanskrit.
[00:16:06.13] So why is the Pythagorean Theorem so important? It's led to our understanding of irrational numbers. It's been used for thousands of years in many cultures, from surveying land to calculating distances. But is it still relevant today? The Pythagorean theorem is implicit in every scientific model or engineering calculation that involves spatial relationships, from quantum physics to Einstein's theory of gravitation. Astronomers at NASA use the Pythagorean theorem, along with other mathematical formulas, to calculate positions of stars and distances to flaring quasars. The Pythagorean theorem is timeless. It's been proven literally hundreds of different ways. And I encourage you to explore more of them.
[00:16:57.26] Hello, and welcome to the teacher's guide segment of this BLOSSOMS module. I'm glad you chose to share this video on the Pythagorean theorem with your students. This lesson teaches about the history of the Pythagorean theorem, along with some proofs and applications. Feel free to use your own motivational ideas and tailor it to your own students. This lesson on the Pythagorean theorem is best for a geometry class where students are familiar with concepts from algebra 1. These activities could all be done individually in pairs or groups, but I think groups of two or three works best. During the Wizard of Oz clip, were students able to spot the three mistakes that the scarecrow makes? There's also an origami proof of the Pythagorean theorem, and a link to that can be found under the For Teachers tab of this lesson. Students may be able to recite ‘ a’ squared plus ‘b’ squared equals ‘c’ squared, but press them for what it means and for how they might be able to prove it.

[00:17:59.17] The wooden puzzle, the three, four, five right triangle, can be replicated on paper, with graphs representing the square units.
[00:18:07.63] For activity number two, give each student or pair of students the worksheet, scissors, and scotch tape. It's important to let the students work on this for a while before giving them any hints. An initial hit might be to combine pairs of right triangles to form rectangles by matching them along the hypotenuse, so that becomes the diagonal of the rectangle. It's much harder for students to figure out how to combine those two rectangles with the remaining square. A later hint could be to combine the two rectangles to form an L shape. Students may be able to get the configuration right, but not know how to interpret it correctly. Another hint might be, where could you place a square relative to the two rectangles to have a square with a side ‘a’ and a square with a side ‘b’? Once they figure that out, or you show them, it may help to draw a boundary separating the two squares. It's important that they share ideas and work cooperatively, and yet not get discouraged. Once it’s explained it's helpful for students to again have time to discuss, since not everyone will understand it at the same rate.
[00:19:22.87] It's important for students to be able to work cooperatively and to not get discouraged. I found that even after it's explained it's helpful for students to have time again to talk, because not all students will understand it at the same rate.
[00:19:37.35] Activity three and activity four should be relatively self-evident. For activity five give students a length of string, a marker, and a scissors. See if they can make a three, four, five right triangle by making 12 even increments. You may need to prompt them that it's OK to use any small increment as a unit. It doesn't have to be inches or centimeters. It could be one hand-width, or the width of a book. It may be helpful to make a major mark on the string denoting three increments, four increments, and 12 increments, to help form the three, four, five right triangle.
[00:20:17.48] For activity number six, the worksheet, if you'd like to do an enrichment activity, you can find Euclid's formula for generating Pythagorean triples under the For Teachers tab.
[00:20:29.20] Thank you so much for sharing this lesson on the Pythagorean theorem with your students. I hope you and your students enjoyed it as much as I enjoyed putting it together.