Pythagoras theorem: proof and applications

Segment one

I am Dr. Kamel Al-Khaled from Jordan University of Science and technology.
Today I will be presenting what was prepared in cooperation with Dr. Ameen Alwaneh in the math module, which is

“Pythagorean Theorem: Proof and Applications”

I hope you will enjoy that.

Before we start our presentation, let us first talk a bit about Pythagoras… he was born in Samos in the year 566 B.C. and died in 479 B.C. He moved from Malta to Phoenicia in the north of Syria then to Egypt, he stayed there for 12 years then went back to Samos – his birth city - he was 65 years old by then . Then he went to Greece and died there.
He was implicated in mathematics and music.

The outlines our module are as follows:

First; I will state Pythagorean Theorem, mathematically and geometrically and in mathematical symbols, then we will solve the right angel triangle. Which means, if we do have lengths of two sides of a right angel triangle, how can we find the third side?

Then we will talk about the converse of Pythagorean Theorem ….i.e. if a triangle satisfies Pythagorean Theorem, does it mean that the angel is triangle? We will see that latter on.

We also will have a question about forming groups of three whole numbers represents a right angel triangle diameter that satisfy with Pythagorean Theorem, then we will draw a proof of the theorem; knowing that there are more than 80 proofs for this theorem. Here, we will be talking only about one of them.
Briefly that is what we will be doing today. I will come back to you in a minute.

Segment two

Dear students …. Let me introduce the following two problems, or two puzzles, or simple questions.
And, I will not give the answers of these two questions; however you will answer them easily on your own by the end of the module.

One of the very well known stories during the Babylonian empire says: if we have a right angel triangle, let us call it ABC, its right angel is B. We where given two gold squares. The length of the first piece equals the side of the triangle call it AB; the second piece equals the length of the side call it BC. So if you where given these two pieces, and have the choice to get a bigger piece of gold that have a length equals the length of the longest

side of the right angel triangle AC. What will you choose two pieces or one piece?
You probably think that two pieces is much better to choose, however, the second choice is that the one pieces side length equals the longest side of the triangle. What is your choice? This may be elusory, but you will be able to answer this question by the end of the module.

The second story: If you live in this house.

Suppose you left the house and forget to take your key along.
The house surrounded by a pavement.
To get key of the out of the house, you noticed that a window is opened, it was 25 feet high. The pavement was 10 feet wide. The question now, how can you reach the window to get the key?

One of the possible answers is to get a ladder and put it on this place, with its lower edge on the side of the pavement, and the other edge is on the lower part of the window. Then we are having a right angel triangle; the height of the window is 25 feet, the pavement width is 10 feet. So this is a right triangle with its legs known, we need to figure out the length of the ladder needed to reach into the window to get the key.

Answering these two questions without enough knowledge in Pythagorean theorem is not easy. I hop you will be able to answer these questions by the end of the module.

The two question; one or two pieces of gold, and finding the ladder's length.
To be able to answer these questions we need to get into the details of the Pythagorean Theorem. Please dear students think about solutions of these two puzzles and hope by the end of the module you will be able solve them.
I will see you in few minutes.

Segment three

Let us find out what is the statement of Pythagorean Theorem? If we have a right angel triangle, say ABC so, the right angel is B, then the theorem says that:

the sum of the areas of the two squares constructed on the two right angel sides AB, BC equals the area of the square constructed on the hypotenuse AC.

In mathematical symbols … we can write the Pythagorean Theorem for any triangle with sides a, b and c …. This triangle is a right angel triangle, according to the above statement, we do have

$a^2 + b^2 = c^2$

This means that the area of this square of c^2 and this square is b^2 and this a^2

If we added the areas of the two squares on the sides a, b we will get the area of the third square that is constructed on the hypotenuse of the triangle with a length of c.

If we look again at the theorem, (see the Graph) we can check the validity of the theorem.
If we have a right angel triangle ABC and its right angel is B. One leg is of length 3

units, a square was constructed on it, and then its area will be 9 units. The other side was length 4 units, a square was constructed on it too, and its area is 16 units. On the hypotenuse which is 5 units the square constructed has an area of 25 units. For the proof of Pythagorean Theorem: let us again remind you about the statement of the theorem, "the sum of the two squares constructed on the sides AB, BC of a right angel triangle equals the area of the third triangle constructed on the hypotenuse of the triangle". If we counted the number of the small square on each side we will find out that they are equal. … Let us count. In the first square constructed in the side AB we have 9 small squares. In the second square BC we have 1,2,3,4, 1,2,3,4 … we have 16 small squares, now according to Pythagorean Theorem the number of small squares in the big square constructed on the hypotenuse AC should be 9+16= 25 and this is really the answer……. Because 1,2,3,4,5, which means that 5 ×5 equals 25 small squares which make the big square constructed on the hypotenuse AC. This proves the theorem geometrically; however by the end of the module we will proof it using different method.

The question now, where do we use Pythagorean Theorem? and what are the applications? This is what we will be seeing in the next segment; please dear students try to think of some applications for this theorem, and feel free to ask any question.

Segment four

Welcome back dear students.
We will now try to" solve the triangle" from our knowledge of Pythagorean Theorem what do we mean by "solve the angel triangle"?

If we have a right angel triangle ABC, its right angel is B, and we have two known sides of the triangle, how we can solve for the third side?
For example, if the length between A, B is 14 and the distance between C, D is 6, how can we find the length of the side AC using the theorem?

We know that according to the Pythagorean Theorem AC square =AB square +BC square

We know AB and BC but the AC is unknown, that is we call it hypotenuse. If we substitute the value of AB which is $(14)^2$, plus 6 square, which is 36, so the answer will be 232, which is the square of the side AC. Back to the statement of Pythagorean Theorem we know that we can construct a square with an area that equals the sum of areas of the two squares constructed on both sides of the triangle which equals 232 square units. But the question is to find the length of AC not the area of the square constructed on it. So to find the length of AC we need to find the square root of 232, the answer is 15.2 units which is the length of AC.

The second example …. We need more space…. If Ahmed lives in this house, he wanted to go the store, knowing that there is a field in the middle. Ahmed has two choices to go from the house to the store; he will either walk along the sidewalk knowing that the distances are 20 and 70 that means he will walk 20 +70 to reach to the corner …. Then walk towards the north with a distance of 50 +70.

So the first choice is that Ahmed will walk staring from the house, and will cover a distance let us call it A1 which is the first choice which means he will walk along the side wake so he will cover a distance of 20+70+50+70=210 meter, i.e. if he walked along the pavement he will cover a distance of 210 meter. Now the question, is there any possibility that he will go to the store using a shorter distance and less time? The answer is yes, he can go directly to the store by walking across the field with the diagonal line connecting the house with the store.

Here we need to use Pythagorean Theorem….. This means that, do we have a right angel triangle? The answer is yes. It is required is to find the distance between the house and the store. …. The triangle sides are 90 m which is 20+70; the other side is 50+70= 120m. As a summary, we have a right angel triangle, with sides 90 and 120. We need to find the length of the hypotenuse using Pythagorean Theorem. Let us call the distance from the house to the store A2, which means the distance from H to S. T. This equals to... Since we are solving for the length not for the area, then we need to find the square root of the first number 90 square plus the second side 120 square. We find the answer 22500. Finding the square root of this number we get 150 meter.

Now dear students, which route is better for Ahmad to use, walking along the sidewalk or passing through the field along the diagonal line from the house and store?

We can find the answer by comparing the numerical values of A1 and A2, we found that the distance in A1 =210 meters while it is 150 for A2. If we asked about the difference and how much distance and time will Ahmed save by using the diagonal walk, we will find that A1 –A2 or 210-150 =60 meter, which means that Ahmed saved 60 meters by using the diagonal path to the store and passing throgh the field corner.

Actually, These are important applications, that can be used in our daily life for Pythagorean Theorem. And the fist question was real application in daily life used of a right angel triangle.

The question now, is the converse of Pythagorean Theorem true?
We will be talking about that in the next segment.

Segment five

Welcome back.
I hope you enjoyed what I have presented so far about Pythagorean Theorem.
The question you may ask, is the converse of the theorem true?

Which means,
If we have a triangle, any triangle, it may not be a right angel triangle- but it the sides satisfies Pythagorean Theorem …. i.e. the area of the square constructed on its longest side equals the sum of the areas of the two squares constructed on the other two sides?

The question is, Can we conclude that it is a right angel triangle?
This is what do we mean by the converse of the Pythagorean Theorem.

Suppose we have a triangle abc and it satisfy this relation $c^2=a^2+b^2$

Does this means that it is a right angel triangle?
The answer is yes, so the converse of the Pythagorean Theorem is true too.
Let us take a real life example.

In constructing a house …. A column of 8 meters was built, and the engineer wanted to find out if the column is perpendicular (on the base), or if it forms a right angel with the base. So in front of his workers, he measured the length of the column, it was 8 meters, and then he fixed a point which is 15 meters away from the column.
So the column's height is 8 meters, the point is 15 meters away from the column. He then connected a rope from the top of the column to the fixed point, he measured the length of the rope, and it was 17 metes. He looked at his workers and thanked them for their accurate work, since he concluded that the column was perpendicular to the earth surface or the base.

Now dear students how did the engineer figured out that his workers where accurate in building that column?
Actually, the engineer used the converse of Pythagorean Theorem; 8 meters is the column's height. The distance of the point sited was 15 meters. The distance from the top of the column to the sited point is 17 meters. He used the following equation $(8)^2$ which is the column's height + $(15)^2$ meters which is the fixed point distance and this is equal to $(17)^2$. Actually this to confirm that means is this equal to this?
So $(8)^2 = 64 + (15)^2 = 225$, and $(17)^2 = 289$, if we added 64 + 225 we will find that it equals 289, then this confirms that Pythagorean Theorem is true, and this means that the column is perpendicular with the base built and the building process is good.

Now if the engineer sited the point at 16 meters instead of 15, this distance 16 meters, the question is how much will this distance be? It is the length the rope the engineer connected the new sited point with the column's top. Certainly it will not be 17 meters; most probably it will not be a whole number, which is what the engineers or people usally don't like to deal with non integers.

So we are always looking for integers. I will give you some time to find out, what is the length of the rope connecting the two points. Knowing that the column's height is 8 meters and the fixed point is 16 meters
In the coming segment we will find out how we can form groups of triples that satisfy the theorem.
We will meet in few minutes.

Segment six

Dear students Welcome back again;
Let us go back to the engineer case where we had a triangle with sides 8, 15, and 17 meters.
The question is how the engineer find out the lengths of the sides of this triangle as integers,

For that, we have the very well known triangle 3, 4, 5 which is the simplest triangle that satisfies Pythagorean theorem with three consecutive integers, and by the way, it is the only case, that satisfy Pythagorean Theorem with consecutive integers.

Let us go back to the engineer's question, … 8, 15, 17, how can we find triangles with whole numbers that will comply with the Pythagorean theorem…. To get triangles with whole numbers there are two methods;

The first is to start with the triplet 3,4, and 5. This as we said complies with the theorem ….

To get triangle with whole numbers, the first way is to multiply this triplet by any number n , let us take n= 7, so for $7\times 3= 21$, and 7 times 4 , for 5 it is 35. So the new sides of the new triangle are 21, 28, and 35. And this complies with the theorem. To check that, If we add $(21)^2 + (25)^2$ it must be equal to $(35)^2$. We conclude that this is one way to get whole number triplets by multiplying the famous triplet (3, 4, and 5) by any number n, where n is whole positive numbers.

So the chosen number is whole positive number n, if the triplet (3,4,5) is multiplied by it, we will get another triplet that complies with Pythagorean Theorem.

So this one of the methods to find a triplet that complies with Pythagorean Theorem.

The other method is … give me any two positive integers, say m, n, where m is greater than n.

We are required to find the sides of the right angel triangle a,b,c as integers, and complies with Pythagorean Theorem. Which is $a^2+b^2 = c^2$. So finding the triplet a,b,c using given values of m and n. We can do that by choosing the value of a to be 2mn, where n and m are already given, so a is 2 multiplied by m*n, also the value of b is m square minus n square,

and as we mentioned before m is greater than n. so m square is greater than n square, therefore, the difference will be a positive number, then b is the difference of m square minus n square. The value of c is $m^2 +n^2$; before we start giving examples on this topic we need to confirm the truth of this relation …. i.e. if we are given any values of m and n where m is greater than n, the values of a, b, and c are as mentioned.

Let us calculate $a^2+ b^2$ …. the answer should be c^2. If we substitute "a square" with its value $(2mn)^2$ plus b^2 which is $(m^2-n^2)^2$, then we simplified the relation we will find certainly

$(2mn)^2 = 4m^2n^2$ …..this is a difference between two squares which is $m^4+n^4 -2m^2 n^2$ as you can see we can simplify these two terms….. Then we can say that this equals m^4 and what is left after subtraction of 2 from this 4 we get $2m^2 n^2$ while n^4 stays as is.

We have now three terms, and the question is, can we have this quantity as a complete square for a specific quantity?

The answer is yes. We are looking for C^2 to be our answer, and it is so. Then this equals $m^2 +n^2$ all Square, if we checked ……. Notice that m^2 all square means m^4 and n^2 all square will result in n^4 and this by this and this by this will give the middle term $2m^2n^2$, and this is nothing but c^2 i.e. giving values for m ,n where m is greater than n and choosing the values of a,b,c .this way will prove the truth of Pythagorean Theorem using this proof.

Accordingly let us take some values of m and n. for example if m is 3, n equals 2, we need to assure that m is greater than n. yes 3 >2 , according to the rules we can take the value of a is 2*3*2 which is 12, that is the length of one side of the triangle, while the value of b according to the rule m^2-n square equals 9 –2^2 which means 4, so the answer is 5, accordingly c= the sum of the two squares which is 9+4=13. Then we can conclude that the triplet that complies with Pythagorean Theorem is 12, 5, and 13.

Similarly we can choose other values for m and n.

I have and idea to share with you, but to start with, I want to choose other values for m and n. to get another triplet. Suppose m=4, n=1. According to this choice the values of a,b,c .

The value of a is 2*4*1 = 8, b = m square- n square =16-1 =15, and c=the sum of the two squares which is 16+1= 17. What does this example reminds you of?
Is it the same as the engineer's example? Was the column's height= 8? Yes. The engineer moved about 15 meter away from the column, the rope connecting the two points was 17 meters; this means that the engineer knows this rule.

Looking at these two examples ……. This triplets and this one, we can see that one of a, b, or c is always multiples of 5, in this example b=5 which is multiple of 5 and here b= 15 which is a multiple of 5 too. This is not a coincidence, but it can be proved, however, we will not be dealing with it here. Also, it is not a must that the side b is the one that is always a multiple of 5 it could be a, or b or c, but it happened here by coincidence.

If we tried to use other values of m and n as in this table; dear students the following table show some values for m and n. in case m=2, n=1 then a, b, c are 4, 3,5; this triangle which we know before. We used this example before, it is m=4 and n= 1 the answer was 8,15,17.
In case of 5,4 the answer was 41,9,40. If we study this table especially the last three columns we will see that at least one of them should be 5 or its multiples. This one this another one and this is another one.. So this a true rule.

Also we can see that the only three serial whole numbers that comply with Pythagorean Theorem are 3, 4, 5 the only triplet with serial numbers.

Then want I wanted to talk about here was how to find triplets of whole numbers that represents the diameters of triangles that comply with Pythagorean Theorem.
Now the important question, how can we proof Pythagorean Theorem? Is there a logical proof? We will talk about it later, but for the time being, I would like you to complete the table by adding 5 more rows to this table

I will come back in few minutes.

Segment seven " the proof segment" .

Hope dear students that you have enjoyed what I have introduced to you about Pythagorean Theorem. And you can form triplets that satisfy the theorem.

The question now …. Is there a proof for this theorem? How can we prove it? Or how can you proof that the area of the square constructed on the hypotenuse equals the sum of the areas of the two squares constructed on the other two sides.

If you browse the net, we will definitely find more than 80 proofs of this theorem. Today we will use the Indian method, which depends on adding areas in different ways.

If we have this square … this square … we divide any side with different ratios… we will call this distance A the rest is called B. we do the same on the other side, the same distance this is A and this is B, and the same for the third side, A, B and the forth side A and B.

If we connect the points A together …. Like this, we will get 4 right angel triangles. One is this triangle, and this and this and this too. What about the part in the middle? Is it square? Yes it is.
Since it is square then it's four sides are equal c, c, c
And C, now we can use this shape to find the area in two different ways:

First using the outer area:

We can divide the area into two parts, the outer area and the inner area
Inside this figure, If we look at the total area, this big square has side of length A+B

then the are of this big square $(A+B)^2$.

Now for the inner area, we can say that the total area is the sum of the five figures areas. 4 of which are right angel triangles…. So we have 4 right angel triangles, the area of each is ½ times the base multiplied by the height. This applies for the second, third and forth triangle. This way we can calculate the areas of the 4 triangles…. We are left with the small square in which its area is c^2 (because the length is c), now if we simplify the equation

Outer area=Inner area
$(A+B)^2=(1/2)\ A\ B + C^2$
…… $= C^2 + 2AB = (1/2) \times 4$
And now $(A+B)^2$ is equal to $2AB+B^2+A^2$
by simplifying this equation and after removing
2AB from both sides of this equation we will come up with
$C^2=A^2+B^2$

If we go back to the figure, we will find that A and B are the legs of a right angel triangle and C represents its hypotenuse. So what we got is $C^2=A^2+B^2$
And in fact this is a proof of Pythagorean Theorem.

Now I will give you the time to brows in the net for some more proofs, since there are more than 80 proofs as I mentioned ,this may be the simplest method for the proof. Some of you may ask why we need this in our daily life, This is what we are going to see in the next segment.

Segment eight.

Dear students …. Hope you enjoyed and learned the way give the proof the theorem. As I mentioned before, you can look for more proofs by browsing the net. …

Let us find out where we can use the theorem in our daily life. What are the applications of this theorem.
I would like to give some examples about this issue… I will not solve them all; I will leave some of these for you as homework.

The First Problem

If we have a lake … this shape, and we need to find its length ….. This is water…. We need to find the lake's dimensions. One idea is to fix two columns, or points …. The first column at A and the second at B … of course we cannot walk on the water, and we would like to find out the distance between the two points A and B. … the idea is to move a specified distance away from point A so that we get a right angel…. Then fix a point, say as C … then connect it with the point B by a thread. By measuring BC and AC We will be able to use Pythagorean theorem to find the distance AB .

In fact according to this, AB is the length which represents the lake's length using the square root …. for the hypotenuse BC^2 that we can modify and control since it is on the land … and represents the length of the tread on the land minus AC square all…. But why is it minus? Because, the unknown is one of legs not the hypotenuse.

According to the theorem this is how to find the dimensions distance of the.
I hope you can think about other
Measurements such as BC= 70m and AC= 28m. And try to find the length of the lake according to the rule. This is the end of the first example.

The second example:

This is an example which commonly seen in newspapers and on TV while reading advertising about TV sets.

They say that a TV set has 19 inch length and its width is 15 inches, we need to find out the number which will be used in the add to tell the diameter of the TV screen. Usually they announce for TV set by using the length of this line (called the diagonal) which is in fact the hypotenuse of a right angel triangle with legs diameter 15, 19.
Then in our announcement we will say a TV set for sale, but how many inches? Let this be X, this the required number.
Then this X, it will be the square root of $(19)^2 + (15)^2$ so the answer is 24 inches. This

how we find a TV diameter and announce it

The third application for Pythagorean Theorem

This is the third example and the last one

If we have the points D, A as show n in the Figure. The distance between A and D is 3 meters, the distance between C and B …… this B and this is C is 5 meters. The question….. Usually this is a real life application. AD represents a column; BC represents another column, at the top of each there is a lamb, we know that light travels in straight lines. We would like to locate a point P where the distance AP + BP is Minimum, given that DC as a distance is 8 meters or units,

… in real life AD is a column of 3 meters height, with a lamb on its top. And BC is another column with its height is 5 meters or units with a lamb on its top. It is required to locate a point between C and D knowing that the distance between them is 8 units. Where is the best location for this point that satisfies the condition that AP+PB is the least possible distance. (Minimum)

One of the possibilities is this ….. This is the new P or it could be this …. Notice dear students that to reach to the solution you need to use Pythagorean Theorem because in both cases we have right angel triangles.
We do not need meter. We can calculate the length for each triangle and add them then get the answer. Where will the point P be at the location required?
This will be your homework.

Finally I wish you have enjoyed what I have presented to you about Pythagorean Theorem.

It seems all of you are able to answer the questions I asked by the beginning of the module. .. Will you be choosing one piece of gold with a diameters equals the hypotenuse of a triangle or two pieces that fit with sides of the same triangle? The answer is clear now, it is the saaaaaame.

They both have the same amount of gold.

As for the house key, what is height of the ladder that we can use to get into the house?

Using simple mathematical process we can answer this question too. Also the lake dimensions and the location of point P are very easy questions now.

One proof of the theory was introduced and a briefing of Pythagoras life was shared. We know that he spent about 36 years of his life in the Arab world …., this is our gratification for us Arabs and that a person like Pythagoras lived among us for 36 years . Finally I wish you all the best.

Teacher's Guide Segment

Dear math teachers, hi, I am Dr. Kamel Al-Khaled. Thank you using this module about Pythagorean theorem proof and applications.

I would like to give you some pints that would help teach this module.
The prerequisites of this module are types of triangles, angle, and some simple calculations.

At the beginning we need to establish a common ground between teacher and students, to point out benefits of the use of Pythagorean Theorem in our life that will lead students to the lesson.

We start the module by capturing the students' attention either by a puzzle, or a joke that shall be answered by the end of the modules.

Show the importance of this topic for the students' life.

Show the student what to be covered during the class period.

Stage of presenting the discussion

General and brief history about Pythagorean.

Statement of Pythagorean theorem, as well as mathematical and geometrical .

Solving the right triangle. Which is finding the length of one side of a right angle triangle knowing the other tow sides

The students may wonder if the Converse of Pythagorean Theorem is true. I hope you draw the attention of the students to solve the question about the engineer during the break and ask them to find the threads length if the distance was 16 meter instead of 15m. Please give help to the students if needed.

Construction of integer right triangles that comply with the theorem, and ask them to fill a table by adding more rows into the given one table

Proof of Pythagorean Theorem, as we mentioned there are more than 80 proofs … ask the students to browse the net for some more proofs

Applications of Pythagorean Theorem, please ask the students to answer all the questions and I will provide the answers.

Finally, The answers of the introduction questions must be answered

Finally, thank you again for choosing this module, hopping you will motivate your students to look for more proofs and applications.
Thank you again.

m	n	a	b	c
2	1	4	3	5
3	2	12	5	13
4	1	8	15	17
4	3	24	7	25
5	2	20	21	29
5	4	40	9	41
6	1	12	35	37
7	2	28	45	53