Rational versus Irrational Numbers

[00:00:00.00]
[00:00:00.36] [MUSIC PLAYING]
[00:00:00.00]
[00:00:28.95] Can you even imagine any civilization without numbers?
[00:00:34.08] Nowadays, can you performed a task, an action, or a transaction without the use of numbers?
[00:00:41.53] The ultimate nature of reality is numbers. This is what Pythagoras, the great Greek mathematician, clearly answered 2,500 years ago.
[00:00:52.39] When we set our own agenda every day, we need to know the time, the hour, sometimes the minutes, even further, the seconds. And all of these are numbers.
[00:01:03.15] If we plan a trip or a visit, we need to make a schedule based on distances in order to arrive at a specific time. We need also to look at the weather forecast and find out about temperatures, which are also expressed by numbers.
[00:01:20.74] Another example is when we proceed with commercial or financial transaction. We need to know about price lists, performing bank transactions, know about interest rates. Therefore, no wonder why we must teach our pupils from early age to use and manipulation of numbers.
[00:01:41.33] Our lesson today is to go in-depth into numbers, which are clearly classified mathematically as either rational or irrational numbers. These two categories of numbers, rational and irrationally, are beautifully combined in geometric objects such as the T squares 45-45, or 30-60 triangles.
[00:02:09.22] The first one combines 1 and square root of 2, and the other 1, 2, and square root of 3.
[00:02:18.67] We have also the circle, or the protractor, that combines 1 and 5. Note also the inner and outer golden spirals that combine in particular the golden number-- 5 equal to 1 plus square root of 5 over 2. And if you were going to actually sequence, 1, 1, 2, 3, 5, and so on. With the n-th element of the sequence being the sum of the two preceding ones.
[00:02:46.90] This beautiful combination of rational and irrational numbers combined in geometrical objects reveals undoubtedly a beauty. Beauty was acknowledged by Proclus 1,500 years ago who said, "Wherever there is number, there is beauty."
[00:03:15.56] [TYPING]
[00:00:00.00]
[00:03:21.42] Hello, my name is Nabil Nassif, and I'm professor in the mathematics department at the American University of Beirut. And this is Sophie Moufawad, a PhD candidate in computer science. We're going now to move to the classroom to talk about rational and irrational numbers.
[00:03:41.31] After our brief production about rational and irrational numbers, now it's time to give a precise mathematical definition about what is a rational number, and what is an irrational number.
[00:03:53.33] So if I take a number x, then that number x can be rational. That is, the ratio of two integers m/n where the denominator n is not equal to 0. And otherwise, if it cannot be put in this form that is a ratio of two integers, then it is said to be irrational.
[00:04:13.62] Now, if I take any number x, that number x can be written as the sum of i plus f, where I is the integral part of the number x and f is its fractional part. And that fractional part f is going to be between 0 included and strictly less than 1.
[00:04:37.71] Now, the strictly f equal to 0, this means that the number x is an integer, which is the same thing as saying that the denominator here n is equal to 1.
[00:04:47.83] To illustrate our definitions, I'm going to ask Sophia to give you some examples.
[00:04:56.75] We're going to start with the first example, which is 48/25. And as you can see, it's equal to 1.92.
[00:05:05.11] Now, to write it as a fractional and integral part, the fractional part would be equal to 0.92. Whereas, the integral part is equal to 1.
[00:05:14.68] Now, I'll give you three minutes to find the fractional and the integral part of the remaining numbers using your calculators.
[00:05:48.79] Note that in the first three cases, the fractional part either terminates, as is 48/25, or is infinite. However, with a repeating pattern, as in 8/3 and 17/7.

[00:06:02.67] When we look into irrational numbers, then we found that there are no fractional part that is terminating or that has a repeating pattern. It looks to us that if we want now to distinguish between rational and irrational numbers, then we must look at the fractional part of each of them, and then try to come up with the results that specify that, for example, for a rational number, the fractional part must be either terminating or must have a repeating pattern.
[00:06:42.98] At that point then, we may define script R, which is the set of all rational numbers that are between 0 and 1. And as well, script I, the set of all irrational numbers also which are between 0 and 1.
[00:06:57.57] If we take the union of R and I, then you get the interval 0, 1. And at the same time, the intersection of R and I is empty because we know that there are no numbers that can be rational and irrational.
[00:07:11.41] So at that point, I would like to pick any number between 0 and 1. And I want to find out the probability that that number be a rational number. And that is the main question of this lesson.
[00:07:31.01] The main question of the module is, if we pick a point F from the segment 0, 1, what is the probability that this number be rational? That is, f belongs to R.
[00:07:57.65] All right, so to answer our main question, then we need some notation and preliminary results. So let me start by taking a number f between 0 and 1. And I want to represent that f in base 10, so I write it as an equation f equal to d1 times 1/10 plus d2 times 1/100 plus d3 times 1/1,000 plus et cetera, plus dk times 1/10 to the power of k. And the di's-- d1, d2, et cetera, dk, they belong to the set 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.
[00:08:36.52] Notation-wise, that equation is written as follows. So f is equal 0. d1, the position of the 10's, d2, the position of the hundred's, et cetera-- dk and so on.
[00:08:53.89] So now we're ready to state the main result that will help us distinguish between rational and irrational numbers. So here's the main theorem. It says the following, that a number f, which is between 0 and 1, represented by 0. d1 d2, et cetera, dk is rational. It's in R if and only if this representation is either terminating or has an infinitely repeating pattern.
[00:09:23.47] This theorem has two parts. One part say that if you are in R, then the representation must either terminate or must have an infinitely repeating pattern. And vice-versa, if this occurs, then the number f must be rational.
[00:09:44.16] To prove the theorem, we're going to go both ways. And that would be starting with the "only if" part. And I'm going to leave it to Sophie to show you how this works.
[00:09:58.26] The proof of the "only if" part of the main theorem is equivalent to two statements. The first statement is, if f has a terminating decimal representation, then f is rational. The second statement is, if f has a non-terminating decimal representation with a repeating pattern, then f is rational.
[00:10:22.11] The proof of the first statement goes as follows.
[00:10:25.66] Now, statement one says if f has a terminating decimal representation, then f is rational. We consider f to be a faction between 0 and 1 that has a terminating decimal representation as follows.
[00:10:39.50] Then, if we multiply f by 10 to the power k, we'll obtain an integer m, which is equal to d1 times 10 to the power of k minus 1 plus d2 times 10 to the power of k minus 2, et cetera, plus dk. This implies that f can be written as m divided by 10 to the power of k, which is a ratio of two integers.
[00:11:02.48] Therefore, f, which is equal to m divided by 10 to the power of k is a rational number.
[00:11:10.84] To illustrate this statement, we will take as an example 0.625, which is a fraction less than 1, and that has a terminating decimal representation.
[00:11:23.25] It is equal to 625 divided by 1,000, which after simplification, is equal to 5 divided by 8, a ratio of two integers.
[00:11:37.51] Without loss of generality, we consider f to be equal to 0. d1 d2 to dk where the repeating pattern is from d1 to dk. The decimal representation of f is d1 times 1/10 plus d2 times 1/100 and so on.
[00:11:55.97] Then, if we multiply f by 10 to the power k, we will obtain d1 times 10 to the power of k minus 1 plus d2 times 10 to the power of k minus 2, and so on, till dk plus f, where the part d1 times 10 to the power of k minus 1 till dk is equal to m, which is an integer. This implies that 10 to the power of k minus 1 times f is equal to m. Therefore, f is equal to m divided by n, where n is 10 to the power of k minus 1. Hence, we obtained a ratio of two integers.
[00:12:35.28] As an example, show that f, which is equal to 0.42851 bar is a rational number.
[00:12:57.90] We take f to be equal to 0.428571 bar. Its decimal representation is shown. Using the same procedure as before, we obtain that f is equal to 428,571 divided by 10 to the power 6 minus 1.
[00:13:17.94] After simplification, we get that f is equal to 3 divided by 7, which is a ratio of two integers.

[00:13:26.45] Thanks, Sophie, for providing us with the proof of the "only if" part.
[00:13:30.32] We're going to go ahead and proceed with the proof of the "if" part. That is, we're going to start with a number f, which is rational. It's a ratio of two integers represented by the sequence of digits d1, d2, d3, and so on.
[00:13:47.49] And from there, we're going to prove that this representation is going to be either finite. That is, supposed to stop. Or, it's going to be non-terminating, but however, with a repeating pattern.
[00:14:01.07] To prove this result, we need two tools. The first one is the Euclidean division theorem. And the second one is the pigeon hole principle.
[00:14:12.04] First tool, the Euclidean division theorem. Statement-- two integers, capital M, which is greater or equal than 0, and capital N, which is greater or equal than 1. Under this assumption, there exists a unique pair of integers d, r such that we have this equality. Capital M is d times capital N plus r.
[00:14:35.27] Or equivalently, if you divide this equation by capital N, capital M/N is equal to d plus r over capital N.
[00:14:45.12] d is said to be the quotient of the division, and r is the remainder of the division of integers from 0 up to capital the N minus 1.
[00:14:56.77] So now we're going to proceed and apply the Euclidean division theorem.
[00:15:01.32] If I take f, which is m/n written as d1 times 1/10 plus d2 times 1/100 plus so on, then if I multiply this equation by 10, I would get 10m over n equal to d1 plus f1. And that f1 is d2 times 1/1 plus d3 times 1/100 and so on.
[00:15:24.98] So in other words, if I consider the Euclidean division of 10 times m by n, d1 would appear to be like the quotient. This is the origin of the algorithm, which is called the successive multiplication by 10 and division by n.
[00:15:43.93] So this is the tale as follows. 10m is equal to d1 times n plus r1. And the remainder r1 will belong to the set of numbers from 0 up to n minus 1. So this is equivalent to 10m divided by n equal to d1 plus f1. That f1 here will be r1 over n, which is d2 times 1/10 plus so on.
[00:16:09.67] So to get the two now, I need to adjust to apply the Euclidean division of 10r1 divided by n, which is-- I do it here. So 10 times r1 is equal to d2 times n plus the new remainder, which is r2. This is a successive procedure. Each of the remainders belong to the set of integers from 0 up to n minus 1.
[00:16:36.08] So the question is, can the successive multiplication by 10 and division by n algorithm terminate?
[00:16:42.63] We're going to break, let you think about it, and come back in a couple of minutes.
[00:17:07.20] Back then to find out the answer of our question, can the procedure terminate?
[00:17:12.39] The answer is yes, whenever you reach a remainder that is equal to 0. At that point, we're going to see that the procedure will terminate.
[00:17:21.21] And if not, the ordered pair of quotient d sub i and remainder r sub i will have to start repeating.
[00:17:33.29] In the case where we reach the k-th division and we obtain rk to be equal to 0, then we will also have that fk will be equal to 0. Therefore, the algorithm would stop at the k-th division. This implies that all the following remainders and all the following quotients will be equal to 0. m divided by n will be equal to 0. d1 d2 to dk. Therefore, we would obtain a terminating sequence.
[00:18:07.68] Let us consider the example of m/n equal to 1/4, whereby m is equal to 1 and n is equal to 4.
[00:18:17.19] By applying the Euclidean division and the successive multiplication by 10 and division by four, we'll obtain that d1 is equal to 2, r1 is equal to 2, and d2 is equal to 5. Whereas, r2 is equal to 0. This implies that 1/4 is equal to 0. d1 d2, which is equal to 0.25.
[00:18:43.86] Let's take two minutes to try this example where we have m divided by n is equal to 5/8 and find its decimal representation.
[00:19:09.69] Back to our exercise. The answer is d1 equal to 6, r1 is equal to 2, whereas d2 is equal to 2, and r2 is equal to 4.
[00:19:19.67] Finally, d3 is equal to 5 and r3 is equal to 0. Hence, 5/8 is equal to 0.625.
[00:19:30.32] Thanks, Sophie. So now we're going to take the case of the successive multiplication by 10 and division by n algorithm. And we will be considering the non-terminating representations.
[00:19:41.11] Now, look into all of these remainders. None of these remainders is 0. Because if it were 0, then of course, we would be in the case that Sophie explained and where the sequence will be terminating.
[00:19:53.13] So here, each of the remainders will be in the set that starts at 1 and ends at n minus 1.
[00:20:02.55] Now, can we prove that there must be a repeating pattern?

[00:20:08.13] At that point, we need the second tool, which is the pigeon hole principle. The statement of this principle is as follows.
[00:20:15.53] If you do have n pigeons, which are supposed to occupy n minus 1 distinct holes, then at least 2 pigeons must occupy by the same hole
[00:20:30.93] Let me take the example of 10 pigeons and 9 pigeon holes. I take the simplest situation, whereby you have 9 pigeons, each of them occupying one of these distinct holes. And the extra pigeon, I'm putting it with the first pigeon in the first hole. But of course, you can have it placed in any one of the others.
[00:20:51.92] In fact, there are several situations. You can have the 10 pigeons, all of them sitting in one single hole.
[00:20:59.66] And I'm going to ask you now a question for this following example. So we have here 3 pigeons, 3 distinct pigeons and 2 distinct holes. And I want to give you two minutes to think about how to place the 3 pigeons in these 2 distinct holes.
[00:21:31.57] Here is the answer to our exercise-- you can have either 2 pigeons sitting in 1 hole and the third one in the distinct hole. Or, the 3 pigeons sitting in one same hole. Of course, you can have a symmetry, whereby you can have these 2 pigeons sitting in that hole here and that one sitting in the first one. Or here, you can have the 3 pigeons sitting in the second hole.
[00:22:09.22] So let us apply the pigeon hole principle for non-terminating sequences.
[00:22:14.77] Let us take, again, the successive multiplication by 10 and division by n algorithm, which we have seen previously. That procedure is not terminating and let me consider the first n remainders, which are r1, r2-- so on. rn minus 1, rn. And I'm going to consider each of these remainder as a pigeon.
[00:22:36.49] Now, we know that the remainders value are from 1 up to n minus 1. So each of these values is going to be a hole.
[00:22:47.00] So here you are with n remainders, n pigeons that are supposed to be taking n minus 1 distinct values, n minus 1 distinct holes. And therefore, by the pigeon hole principle, I can say that there exists at least two remainders, rj and rk, out of these n remainders, such that rj is equal to rk.
[00:23:15.69] As a consequence of applying the pigeon hole principle, then we have found a pair of integers j, k, such that j is less than k, and both are between 1 and n. And with rj is equal to rk.
[00:23:30.50] Then, we can say that 10rj is dj plus 1 times n plus rj plus 1. The quotient is dj plus 1 and rj plus 1. And similarly, division of 10rk by n will give us the quotient dk plus 1 and the remainder rk plus 1. Since the Euclidean division provide a unique pair of quotient and remainder, and since rj is equal to rk, then this implies that dj plus 1 is equal to dk plus 1 and rj plus 1 is equal to rk plus 1. And this can be repeated more generally to obtain that dj plus l is equal to dk plus l and rj plus l is equal to rk plus l where l is between 1 and k minus j.
[00:24:22.27] So by this argument of recurrence, we can say now that the rational number m/n is just 0. d1 d2 up to dj. And then, from dj where the two remainders rj and rk to be equal. From that point on, then we will be having the repeating pattern, which is dj plus 1 up to dk. And the length of this pattern is precisely k minus j. And this k minus j is exactly between 1 and minus 1.
[00:24:57.10] Previous argument to the fraction m/n where m equal to 6 and n equal to 7. So it gives the following results from d1, r1 up to d7, r7. And all of the remainders from r1 on belong to the set of integers from 1 up to 6 since we are dividing by 7.
[00:25:22.21] Now, look here that r1 is equal to r7. And this is an illustration of the application of the pigeon hole principle. You can see it here, we have the first remainder is 4, the second is 5, the third is 1, so on. The seventh is 4. And we have 6 holes because we have 6 values that can be taken by the remainders. So therefore, we must have at least two pigeons that will occupy the same hole. And in that case, it is r 1 and r7. And all the others are being placed in the other holes-- 1, 2, 3, 5, and 6. So 1, 7 is the first pair such that r1 is equal to r7. And therefore, we can say that 6/7 is 0. d1.
[00:26:08.80] Then, the repeating pattern is from d2 up to d7. And this would give us 0.8571428 as being the repeating pattern. And the length of this pattern is 6.
[00:26:29.30] Now, take two minutes to find the decimal representation of the fraction m/n where m is equal to 2 and n is equal to 3.
[00:26:52.73] The solution of the exercise goes as follows. From the first division, we obtain that d1 is equal to 6 and r1 is equal to 2. Then, d2 is also equal to 6 and r2 equal to 2 and so on. Therefore, we can conclude that r1, r2, r3 are all equal. And hence, they all lie in the same pigeon hole. Thus, 1, 2 is the first pair such that r1 is equal to r2.

[00:27:19.31] Therefore, 2/3 is equal to 0.66 bar. And the length of the pattern is 1.
[00:27:30.71] So let us be back now to the main question of the lesson. I remind you that we have the set of script R, which is the set of all rational numbers that are between 0 and 1. And script I, which is the set of all irrational numbers that are between 0 and 1.
[00:27:52.02] And that I pick any number at random from the interval 0 and 1 and I'm asking the following question, what is the probability that this number is irrational one? And that was the question.
[00:28:09.57] To answer this question, we note first that both of these sets are infinite sets. That is, the number of elements that are in each set is infinite. Let me put here the absolute value of R as the number of elements in R. And I'll put it here as infinity. The same thing goes for I. But the issue here is that this infinity is distinct from that infinity.
[00:28:44.92] We're going to examine know how we can distinguish between these two infinities, the infinities of each of the sets R and I. Which one of these two infinities is bigger?
[00:28:57.60] If f is a rational number, then f is one of these two forms, while if f is irrational, then f has an infinite representation with no specific pattern. Hence, you can guess that there are much more ways to obtain elements in I than elements in R.
[00:29:14.18] Moreover, we're going to show that R is countably infinite. To understand this concept, we're going to define R sub n, which is the set of all rational number of the form m over n plus 1 where m is equal 1, 2, up to n, with the greatest common divisor of m and n plus 1 being 1.
[00:29:33.35] Example of R sub n for n equal to 1. You have R sub 1, which is 1/2. Only one element. For n equal to 2, 1/3, 2/3. For n equal to 3, n equal to 4, you can see what are R sub 3 and R sub 4. And for n equal to 5, I'm giving you two minutes to find out what is R5.
[00:30:08.78] It is R5 consisting of two elements, R10 and R11. We can, therefore, enumerate all the elements or R as R sub 1, R sub 2, R sub 3, R sub 4, implying that the set R, set of all rational numbers, is countably infinite. That is, there is a one to one relation between R and the set of natural integers N.
[00:30:32.83] On the other hand, the set of irrational numbers I is unaccountably infinite. And this follows from the fact that f is international if and only if it's infinite representation has all its elements belonging randomly to the set 0, 1, 2, up to 9. And at that point, the proof of uncountability of I can be obtained using Cantor's proof by contradiction.
[00:30:56.39] The proof by contradiction goes as follows. Let us assume countability of I. That is, the elements of the set of irrational numbers can be enumerated as i sub 1, i sub 2, et cetera.
[00:31:07.76] Now, we express each element in this sequence using its decimal representation as indicated by this matrix. But now, let us define I bar equal to 0. f1,1 bar f2,2 bar, et cetera. Such that each f1,1 bar, f2,2 bar are randomly selected and satisfy f1,1 bar different from f1,1, f2,2 bar different from f2,2 et cetera.
[00:31:36.20] So here we get the contradiction, two facts that contradict each other. On one hand, i bar is irrational. It belongs to I. But on the other hand, i bar is not in the original list that you have assumed i sub 1, i sub 2, i sub 3, et cetera. That is, I is uncountably infinite, contrary to R, which is countably infinite. So what we have concluded now is that we have two infinities-- one for the set of rational numbers, which is countable, and one infinity for the set of irrational numbers, which is uncountable. And that second infinity is a bigger infinity than the previous one.
[00:32:21.29] All right, now both Sophie and myself are ready to conclude.
[00:32:26.09] OK, so as we have seen, the number of elements in R is countably infinite, and we denote it by aleph 0.
[00:32:34.45] And the number of elements in I is uncountably infinite, and we denote it by c.
[00:32:41.28] Hence, c is much bigger than aleph 0.
[00:32:44.45] And therefore, the probability to pick a number from the interval 0, 1 and have this number belong to the set of rational number is aleph 0 over c. And that is a very small number.
[00:32:57.80] So here we are at the end of our lesson. We thank you very much, and we look forward to see you sometime in the future for other lessons on numbers or other topics of mathematics.
[00:33:21.21] Teacher's Guide for BLOSSOMS lesson Rational versus Irrational Numbers.
[00:33:27.82] Prerequisites for the lesson.
[00:33:30.59] The students should be aware of some definitions in discrete mathematics, such as sets, unions, intersections, logical definitions of "implies that" or "if then" and equivalences "if and only if."
[00:33:47.78] Some knowledge of elementary probability definition and one-to-one mappings or bijections.
[00:33:56.29] Tools. To complete the lesson, a calculator is needed. The calculator's functionalities should include the square root function and the number pi.

[00:34:08.20] Remark: Note that, nowadays, smartphones' calculators include those functionalities.
[00:34:16.17] Concept. The goal of this lesson is to teach students how to distinguish in mathematics between rational and irrational numbers.
[00:34:29.03] Overview. In this video lesson, the activities will take the students through a process that allows them to find out the strict digits representation of the fractional part of a rational number, that can be either finite or infinite with a repeating pattern.
[00:34:47.48] In contrast, the fractional part of an irrational number has an infinite representation with randomly distributed digits.
[00:34:57.38] The video lesson spans about 32 minutes and provides 7 exercises for students to work out in groups and in consultation with their classroom teachers.
[00:35:10.41] The exercises exclude the main question of the lesson, which should trigger the student's interest in the lesson.
[00:35:18.14] The entire duration of the video demonstration and exercises should take about 50 minutes. Or equivalently, one classroom session.
[00:35:35.02] After defining rational and irrational numbers, the students are asked to solve the first exercise.
[00:35:42.73] The first exercise of the lesson is in the form of six cases, which should take, at most, five minutes. The purpose of this exercise is to learn how to extract the integral and the fractional parts of any number. And consequently, distinguish visually the difference between rational and irrational numbers by looking at the fractional part of each case.
[00:36:09.41] Note that the first three cases of the exercise correspond to rational numbers, whereas the last three correspond to irrational numbers.
[00:36:20.35] Given that calculators have limited rounded digit spaces, the instructor should help the students to obtain corresponding infinite representations, particularly in the first three cases of rational numbers.
[00:36:36.63] The instructor should meticulously state the main question of the lesson and afterwards he/she is recommended to invite the students to think of a probable answer and write it on a piece of paper.
[00:36:52.66] Then, the instructor must make sure that the students must understand that the representation and notation of the fractional part of a number, since the rest of the lesson is totally dependent on these notions.
[00:37:09.29] The instructor is recommended to present the two parts of the main theorem, the "if" part and the "only if" part while stressing on the notion of logical equivalence "if and only if."
[00:37:23.48] Then, the plan and the proof of the "only if" part is presented, and the instructor should stress that the proof includes two cases or statements. The first is that of a terminating fraction, while the second is distinctly that of a non-terminating sequence with a repeating pattern.
[00:37:44.48] The proof and example of statement 1 are straightforward.
[00:37:49.21] In the proof of statement 2 of the "only if" part, the obtention of the identity resulting from the multiplication of f by 10 to the power k should be stressed and illustrated in Exercise 2.
[00:38:03.44] The proof of the "if" part is first based on the Euclidean division theorem. The instructor should stress this concept in particular the notion that the remainder belongs to a finite set that includes zero. The procedure is self-explanatory in the video and should enable the students to answer the question regarding determination of the representation of the rational numbers. The students are given 3 minutes to solve Exercise 3.
[00:38:34.72] The answer to Exercise 3, in addition to an example of a terminating sequence representation should allow students to complete the exercise that follows.
[00:38:47.05] Exercise 4 illustrates the proof of the "if" part and should be completed in a couple of minutes.
[00:38:53.63] Then, the instructor must insist on the fact that infinite representations of rational numbers result from the fact that all remainders in the successive multiplication by 10 and division by n algorithm are never zero and takes values between 1 and n minus 1.
[00:39:14.49] At this point, the instructor assists the students in the understanding of the pigeon hole principle and the example of 10 pigeons and 9 pigeon holes.
[00:39:26.81] Exercise 5 of the 3 pigeons and 2 holes is a very straightforward application of the pigeon hole principle.

[00:39:35.66] The application of the pigeon hole principle to the non-terminating sequences is probably the most abstract part of the lesson and consequently the hardest part. In case the mathematical level of the students is not sufficient for understanding the general proof, the instructor can omit that proof and limit the explanation to the 6/7 example.
[00:40:01.62] Exercise 6 should be easily done by the students and the example of 6/7 could be conducive to understand the general proof.
[00:40:12.10] The introduction of the main questions answer is intuitive and should allow the students to perceive that there are 2 infinities, one for the rational numbers and another for the irrational numbers with the second one much bigger than the first infinity. The rigorous proofs of these facts are based on the countability of the rational numbers and the uncountability of irrational numbers.
[00:40:40.90] To obtain countability of the rational numbers, a sequence of subsets Rn is introduced. Examples of sets Rn are given for n equal 1, 2, 3, and 4.
[00:40:54.83] In Exercise 7, the students should be able to find R5, which is 1/6 and 5/6 since in 2/6, 3/6, and 4/6, the numerators and the denominators have GCD greater than 1.
[00:41:11.67] The students should be assisted in understanding that by defining R through the rn's, he/she obtains a one-to-one relation between R And the set of natural numbers.
[00:41:25.90] The proof of uncountability of the set of irrational numbers, making this set a much bigger one than the set of rationals, is based on Cantor's "diagonal" argument.
[00:41:37.96] Finally, the answer to the main question of the lesson is obtained through the fact that the probability to have a number f picked at random between 0 and 1 is the ratio of the number of rationales over the number of irrationals, n1 over n2, with n1 being much smaller than n2. And this is a very small number.
[00:42:05.29] [MUSIC PLAYING]