

Amazing Problems: Arithmetic and Geometric Sequences

Peace be upon you, and welcome to this lesson. My name is Abbas Al-Hamada, a mathematics teacher at Al-Saudia Middle School. I believe that mathematics is wonderful and fun, and today I have some amazing problems for you that we will solve together.

Are you ready? Alright, solve this problem:

I want you to find the sum of the numbers from 1 to 100. Can you do that within two minutes? Try it, and I will get back to you shortly.

Welcome back, maybe you were not able to complete the calculations, but it is alright...

This problem falls under the title arithmetic sequences or patterns. Studying sequences is essential in statistics and engineering, since they provide quick solutions for problems that require - without relying on the laws of sequences - a long time, so they are useful to speed up the calculations.

Simply, a sequence is a series of related elements, such as the element is equal to the previous one plus or multiplied by a variable number or a constant.

We can define the sequence of real numbers as a real-valued function whose domain consists of the positive integers, and its corresponding range is the set of real numbers.

Now, work in groups to discover the mathematical pattern of one of the following sequences, and then write the next element in the sequence:

1, 4, 7, 10,

1, 2, 6, 24, 120,

2,2,2,2,

0, 1, 1, 2, 3, 5, 8, 13,

Welcome again, I hope that you may have noticed that in the first series, each element was equal to the previous element plus 3, and thus the next element after 10 is 13.

In the second series, the first element was multiplied by 2, and the second element was multiplied by 3, and the third by 4, and so we have to multiply 120 by 6 to get the next element which is 720.

The third series is an easy one, all its elements are similar and equal to 2, and the next number is also 2.

Regarding the last series, the first two elements were unchanged, then each element is the sum of the two previous elements, and this series in particular has several mathematical applications. It is called the Fibonacci series and there is an interesting lesson on the BLOSSOMS site that covers this Series titled "Amazing Drawing Patterns and the Differential Equation" presented by Laura Zager. You can watch it after we finish this lesson.

The first series and others with similar patterns are called arithmetic series since the difference between any two consecutive terms is constant, which is called the common difference and given the symbol d .

We can express this by $d = a_{n+1} - a_n$

We can also discuss this part as follows:

$$a_1 = a$$

$$a_2 = a + d$$

$$a_3 = a + 2d$$

$$a_4 = a + 3d$$

$$a_n = a + (n-1)d$$

The last formula is called the general term of an arithmetic sequence.

But what if we did not know the first term, and we know the term a_k , then $a_n = a_k + (n-k) d$

Now, let's do this activity:

Find the 10th & the 100th term for the arithmetic sequence

5, 7, 9,

Welcome back. Were you able to find the answers? Where $a = 5$ and $d = 7-5 = 2$

$$\text{Now, } a_{10} = a + 9d = 5 + 9 \cdot 2 = 23$$

$$\text{And } a_{100} = a + 99d = 5 + 99 \cdot 2 = 203$$

Furthermore, if we want to know the term whose value is 45, then

$$a_n = a + (n-1)d$$

$$45 = 5 + (n-1) \cdot 2$$

$$n-1 = 40 / 2$$

$$n = 21$$

The 45th term has a value of 45.

Let's elaborate and explore these patterns. Can't we write the sum of these terms like this, assuming that the sequence is finite?

$$S_n = a + (a + d) + (a + 2d) + \dots + (m-2d) + (m-d) + m$$

We can also write them in reverse, as follows:

$$S_n = m + (m-d) + (m-2d) + \dots + (a + 2d) + (a + d) + a$$

Let's add the two equations, as follows:

$$2S_n = a + m + (a + m) + (a + m) + \dots + (a + m) + (a + m) + (a + m)$$

Or $2 S_n = n (a + m)$ and from this formula, we find

$$S_n = (n / 2) (a + m)$$

This is a great result, but what if we did not know the last term 'm' and we were given the common difference of the sequence instead?

Since m is the last term, and is equal to:

$$m = a_n = a + (n-1) d$$

Let's substitute the value of m in the last formula

" $S_n = (n / 2) (a + m)$ ", we find:

$$S_n = (n / 2) (a + a + (n-1) d)$$

$$S_n = (n / 2) (2a + (n-1) d)$$

I'll leave you now to find the sum of the numbers 1 to 100 within two minutes based on what we have learned, remember in just two minutes, and the sum of the first fifty numbers of the arithmetic sequence: 2, 4, 6, 8, 10, ... in two additional minutes. You can do it, dear students. I will meet you shortly.

Welcome, it must have been very easy for you. Simply, all what you had to do was to use the formulas and substitute the values you have been given.

$$S_{100} = 100/2 * (1 + 100) = 50 * 101 = 5050$$

$$S_{50} = (50/2) * (2*2 + (50-1) * 2) = 25 * (4 + 98) = 2550$$

And now, see what happened to me yesterday when I was playing chess with my friend.

Abbas: This game is very stimulating to the mind, and make you think deeply before each movement.

Abdullah: I certainly agree with you. There is a funny story behind the invention of this game. It took place in India long time ago.

Abbas: In India! What is it?!

Abdullah: A man invented this game and presented it as a gift to the king, who was very delighted, and asked the man to choose a reward for his work.

The man requested to be given for each square of the sixty-four squares, a grain of wheat for the first one, two grains for the second, four grains for the third and eight for the fourth and so on to double number of grains up to the sixty-fourth square.

Abbas: What a strange request this is!!!

Abdullah: The king thought the man asked for a simple thing, and ordered his minister to give the man a bag full of wheat.

Abbas: What happened next?

Abdullah: The man asked to be given exactly the amount he asked for. The consultants of the king were engaged to calculate accurately the exact amount of wheat, and found out that it is very huge and cannot be afforded by the king and exceeds the production amount of wheat for all the kingdoms for several years.

Abbas (Looking at the students): Is this story reasonable?

Do you think this story makes sense?!! Think about that for few minutes and try to calculate the amount of wheat, and I will get back to you soon.

Welcome back. Let us examine this type of sequence:

1, 2, 4, 8, 16, 32,

It is a geometric sequence in which each term equals the product of the previous one with a constant real number called the common ratio of the sequence, which is given the symbol r , and in this case $r = 2$.

This could be written as:

$$a_1 = a$$

$$a_2 = ar$$

$$a_3 = ar \cdot r = ar^2$$

$$a_4 = ar^3$$

But, if the first term is a_k , then $a_n = a_k r^{n-k}$

What about the total number of wheat grains in the chess story?

$$S_n = a_1 + a_2 + a_3 + \dots + a_n$$

$$S_n = a + ar + ar^2 + ar^3 + \dots + ar^{n-1} \quad (1)$$

If r is not equal to zero, by multiplying both sides of the equation by r , we get:

$$rS_n = ar + ar^2 + ar^3 + \dots + ar^n \quad (2)$$

By subtracting the first equation from the second equation, we get:

$$rS_n - S_n = ar^n - a$$

$$(r-1) S_n = a (r^n - 1)$$

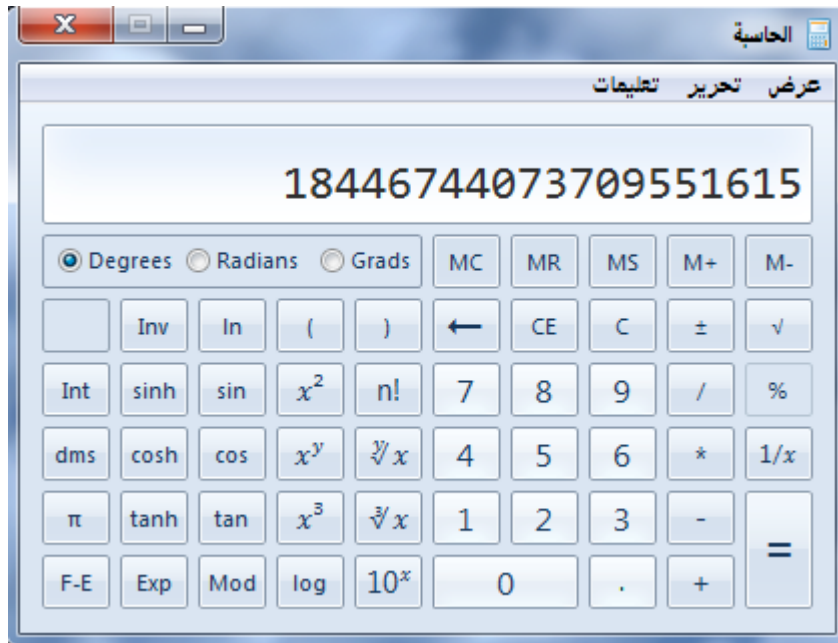
Note that from equation (1) If $r = 1$, then

$S_n = a \cdot n$, and if $r \neq 1$, then

$S_n = a (r^n - 1) / (r - 1)$, and because there are 64 squares in a chessboard, then

$$S_{64} = 1 (2^{64}-1) / (2-1) = 2^{64} - 1 = 18446744073709551615$$

(Eighteen Quintillion, four hundred forty-six Quadrillion, seven hundred forty-four trillion, seventy-three billion ,seven hundred nine million, five hundred fifty -one thousand, six hundred fifteen grain of wheat)



Suppose that every 1000 grain of wheat weighs 40 grams, the estimated weight of wheat in the story will be more than 700 billion tons, equivalent to 1,000 times the annual world production of wheat.

Well, after we have seen how quickly numbers are multiplied in geometric sequences, I'll give you this offer, I have cars for sale on monthly installments for a month or 30 days only.

You pay on the first day one halala, two halalas on the second day, four halalas on the third day, eight on the fourth and so on until you reach the thirtieth day. I will give the one who completes the thirty payments a modern luxury car!!! In addition to that, I will deposit a million riyal in his account at the bank!!!

If you would like to buy a car or if you have any further questions about this lesson, you may send them to my e-mail

abs@mohaffez.com

Peace, mercy and blessings of God