The Power of Exponentials, Big and Small

Nataly, I just hate doing this homework.

I know. Exponentials are a huge drag.

Yeah, well, now that you mentioned it, let me tell you a story my grandmother once told me about exponentials. There used to be a queen in India, and she got really bored playing the routine games. And so she asked all the mathematicians in the country to come up with a new game to amuse her.

And there was a poor mathematician who after years of carefully working through different ideas came up with the game of chess. And the queen, she was so pleased, and asked the mathematician to name her price.

Tell me, my math wizard, any reward you feel worthy of. I love your game of chess.

My lady, I'm a poor woman. All that I need is to have enough grains of rice to feed my family. I would like to have one grain of rice for the first square of the chessboard, two for the second, four for the third, and so on. Every square must have double the number of grains as the previous one.

Ha. A chessboard has only 64 squares. That's a triviality for a mighty queen like me. Treasurer, please give this young mathematician all the grains she's asked for.

The queen seemed to think that the award was very small, a triviality, and that she would be able to pay the mathematician with a single bag of rice. Do you think the award was very small?

Hm. I think I need some help here. Do you think the award was very small? Was the treasurer able to pay off the mathematician with a single bag of rice? Was the queen able to pay off the mathematician at all? Discuss with your neighbors and with your teacher, and we'll be back in a few minutes.

My lady, I think there is a problem.

What is it?

Well, we've used hundreds of bags of rice, and we have not even covered half of the squares of the chessboard.

How can that be? We just started with one grain of rice.

Well, as you can see, at the beginning, we just needed one grain of rice. And then three and seven. Each time we added the rice for a new square, we added one more grain than the amount on all of the previous squares combined.

On some of the early squares, I noticed that the total we had paid out up to that square was 2 to the n minus 1 for the square n. For instance, we paid 2 to the third minus 1 or seven grains up through square three. Since we added eight more on square four, we had paid a total of 2 to the fourth minus 1, or 15 grains through that square.

Through a technique called mathematical induction, we can show that we paid 2 to the n minus 1 grains through the nth square no matter the value of n.

But what caused us to pay so much rice. Even though we doubled from the previous square, we have only 15 grains of rice til now.

Well, the problem, Your Majesty, is that after the first few squares, our payouts grow very rapidly. While at the end of the first row, we've only placed down a few hundred grains, and by the second, we've still placed down less than a full bag.

We reached a million grains paid by the third row, and this is when the payments started getting big quickly. We'd already placed down a billion grains of rice, more than 1,000 sacks by the fourth row when we still had more than half of the board to go.

Had we been able to keep bringing the rice, we would have paid a trillion grains by the fifth row, a quadrillion by the seventh row, and according to my calculations, more than 18 quintillion grains of rice in total. That heap of rice would have been larger than the largest mountain in the world.

Curse you, wise mathematician, you tricked me. I've gone from the richest queen in the world to the poorest woman.

See, ignore exponentials, and you could lose a fortune.

Well, that make sense. But I'm not a queen, and I have no mathematicians in my employ, so I'm still stuck with my homework.

Guys, guys, oh my, let me tell you. I just made a fabulous discovery.

Hey, John, what's going on?

Nataly, let me tell you. So I found when you take a stack of paper and you tear it in half and put one half on top of the other, it becomes twice as tall with each tear, which means in no time flat this pile is going to be enormous, very tall. So let me tell you my friends, today, I am going to the moon. See you.

Oh geez, John must have had too much coffee today.

It seems strangely similar to the one our queen faced in the story. There with every new square, the amount of grain doubled, and here with every new tear, the height of the stack doubles. So since we know exponentials grow so fast, you never know, starting with a 0.1 millimeter thick sheet, you might actually reach the moon.

I don't buy that. The moon is too far away. Let's think of a height of a person first.

What do you think? How many tears would we need to reach the height of a person? How many tears would John need to reach the moon? Do you think we should call NASA with our new discovery? Think about it a little bit, and we will come back shortly.

Just as we suspected, the pile grows taller in a hurry. Because we're doubling the number of sheets of paper every time after n tears, I guess we could say we have 2 to the n sheets of paper.

So if I assume the height of a human being is 1.75 meters, and knowing that there are 10,000 sheets of paper in a meter, help me figure out that an average human being is about 17,500 sheets of paper tall, which is 1.75 times 10 to the power of four in scientific notation.

So checking my powers of 2 table, it looks like my stack will be roughly 1.64 times 10 to the fourth sheets high after 14 tears, which is not quite the height we need. However, after 15 tears, the stack will be 3.28 times 10 to the fourth sheets tall about the height of your classroom and certainly taller than anyone I know.

So 15 tears wasn't much to get to the ceiling of the classroom. Not bad, we go to the ceiling in just 15 tears.

For sure, but when I looked at the moon, things started getting really crazy. The moon is around 3.84 times 10 to the eighth meters from Earth. And I again used the fact that there are 10 to the fourth sheets of paper in a meter to determine that the moon is about 3.84 trillion sheets of paper from the Earth.

Seems like a lot, but after checking my handy power table, I saw that after 41 tears, we will have 2.2 times 10 to the 12th sheets. And after 42, we will have 4.4 times 10 to the 12th, more than enough to reach the moon.

42 tears, wow, John must have already made it to the moon. Let's go and check on him.

Ah man, tearing paper is so hard. I've only torn it five times, and I just can't tear it anymore. I'll never make it to the moon at this rate.

Boy, he gives up really fast.

Well, I think as the stack gets taller and taller, it becomes harder and harder to tear it in half.

Yeah, plus I guess when we tear the paper into half each time, the area the stack covers decreases exponentially. I wonder if we'll be even able to see the stack after 42 tears.

Oh, now I see why we can't tear it 42 times. Well, I'm not a queen. We can't make it to the moon. I wonder what exponentials are good for.

Hmm, well, let me show you a neat trick I learned when I was a little kid.

Let's go.

Hey, John.

Hey, John.

Oh hey. How are you guys? Geez. Tootsie rolls, I love these things.

We got you 10,000 tootsie rolls.

10,000 tootsie rolls.

You know, because you could not make it to the moon.

Oh my goodness. Guys, tootsie rolls are my absolute favorites.

Yeah, we'll give you 10,000 every day of the month.

10,000 tootsie rolls every day of the month. Swati, what do I have to do to get 10,000 tootsie rolls every day of the month?

You just have to give us one today, double tomorrow, double the next day, and so on, for a month. You know just so that we have a little bit for ourselves.

Yeah, you're on. 10,000 a day. Thanks, guys. Here's your tootsie roll for today. Have a good one.

Thank you.

Geez, I really do think exponential growth is pretty quick. And I bet we're going to end up having more tootsie rolls than John by the end of the month.

On what day of the month, would we have given each other the same number tootsie rolls? And how many more tootsie rolls should John have asked us for in order to have the same number of tootsie rolls by the end of the month? We'll give you some time to think it through, and we will come back soon.

Wow, Swati, you really knew what you were doing. I actually drew a plot of the tootsie payments by day n versus John's tootsie roll total payments. And even though, originally, the gap between the payouts seemed to be increasing, it closed up, and you had broken even by day 18.

Just like with the grain payouts to the Indian queen we discussed earlier, you had been paid 2 to the power n minus one tootsie rolls by the nth day. And by the last day of the month, you had pulled in a whopping total of one billion tootsie rolls.

Dividing by 30, John should have asked for 35 million tootsie rolls a day to have broken even.

Exactly, now this is what you call a lifetime supply of tootsie rolls. We have over a 3,000 metric ton of tootsie rolls in our storage now. Well, it's pretty clear from the graph that you ended up gaining tootsie rolls much faster than John. What would have had happened if the amount John had received by day n had been 10,000n squared, instead of 10,000n.

Surely, John would be getting more money overall, but does the function 10,000n squared grow faster or slower than our exponential payout.

Hmm. What do you think? On what day of the month would we have exchanged the same number of tootsie rolls as John? What would have happened if we had paid John 10,000n to the 50 by the nth day, which means by the second day, we would have paid him 11 quintillion, that's 11 billion billion. And by the third day, we would have paid him seven octillion. That's 7 billion billion billion by day three.

Do you think we would have ever caught up with the doubling for the scheme of ours? We'll leave you to discuss now, and we'll be back shortly.

Well, nothing solves a problem like a good old fashioned graph. I plotted two curves, one of which is 2 to the power n minus 1, the number of tootsie rolls we have been paid by day n. The other is 10,000n squared, the number we paid John by day n. Sure enough, we still end up making more tootsie rolls by the end of the month, even though we are paying John a lot more tootsie rolls in this new scheme.

Though paying John seven quintillion tootsie rolls by day three is a pretty scary concept, I made another plot to see if we end up with more tootsie rolls than him with the n to the 50 scheme. Here, I compare 2 to the power n minus 1 and 10,000n to the 50. And sure enough, we eventually end up getting paid more.

It took over a year before we made as many tootsie rolls as we paid John. But as you can see, in the long run, the exponential is growing much faster than even the curve 10,000n to the 50.

To be fair though by the time the two payouts meet, the amount of tootsie rolls we would have exchanged would be more than the weight of the total observable universe. It's a freaking 137 digit number.

True, but in general, exponentials with base more than one would eventually become bigger than polynomial functions in general.

How do you figure that?

I noticed something interesting when dividing the payouts between consecutive days for each of the payout functions. When I have an exponential payout function like 2 to the power n, or more generally, like b to the power n with b strictly greater than one, and when I took the ratio between the nth and the nth plus first element, I always got the same value b no matter what the value of n was.

That makes sense. It's like saying we doubled the number of tootsie rolls every day no matter what day we are on.

But when I look at a polynomial function, like to the power 50, or more generally, n to the power k, I noticed that when I took the ratio of the n plus 1 to the power of k and n to the power of k, I got n plus 1 divided by n, the total to the power of k, which can also be written as 1 plus 1 over n, the total to the power of k.

As n approaches infinity, this goes to one. So once n gets big enough, we are sure that our exponential function is growing faster than our polynomial function regardless of how big the power is on the polynomial or how close to one the base is on the exponential.

Intriguing, isn't it? Since in the long run, we multiply consecutive numbers by a larger value in an exponential function than we do in a polynomial function. We conclude that the exponential will eventually exceed the polynomial function, even if it takes a long time as in the case of our 10,000n to the power 50 example.

In other words, exponential functions always grow faster than polynomial functions in the long run.

Exactly.

Guys, guys, so first just going to say I may be a tiny bit annoyed that recently I had to pay you guys maybe like 30,000 metric tons of tootsie rolls, but I just learned the coolest thing in computer engineering class.

Oh, what?

Let me tell you. So apparently, there's a law called Moore's law, which states that the number of transistors on a square inch of a computer chip tends to double every two years.

Oh, I guess this means that the size of a computer with a given amount of computing power is expected to be halved after two years.

You know recently, in my history of computing project, I was reading about this computer called Cray-1 supercomputer, which was released in 1976. Based on our knowledge of exponentials, if I halved it's size every two years, I'm sure it would fit on my desktop before too long.

But computers were so slow back then.

Well, fine, let's pretend we have 25 Cray-1 supercomputers instead.

So what do you think? Do you think that something with the same computing power today could fit in your classroom? Work it out with your neighbors and your teacher, and we'll be back in a little bit.

Hey, that was a pretty cool calculation. It was just like our rice, and the queen, and the tootsie rolls. Except that here for computer chips, we halved the size each time instead of doubling it.

Whoa, let me tell you a crazy thing about this is that those 25 Cray-1 supercomputers have about the same computing power as a single iPhone 4 does today.

Wow.

Which means Moore's law really did a pretty good job of predicting the size of that iPhone.

Hmm. I must say I'm getting really wound up for exponential functions. Just like exponential growth exponential decay, as in the case of Moore's law, is extremely rapid. So it seems to me that exponential functions are just about the fastest growing and the fastest decaying functions in the world.

Not really. I mean I'm sure I can write a function that grows faster than an exponential

Really?

Yeah, consider factorials for example.

N factorial is equal to n times n minus 1 times n minus 2 all the way down to 1. For instance, 4 factorial is 4 times 3 times 2 times 1, which is 24. The result is something that grows very quickly. 10 factorial, for example, is 3,628,800. 20 factorial is already more than 2.4 times 10 to the 18.

Meanwhile, 2 to the 10 is just 1,024. And 2 to the 20 is just over a million, which are much smaller numbers. If you look at the ratio between two consecutive factorials, n plus one factorial is n plus 1 times larger than n factorial, while 2 to the n plus 1 is only two times larger than 2 to the n. So the ratio between factorials is increasing as n increases, while the ratio between exponential values is staying just the same.

Well then, I guess it certainly stands to reason that the factorial function should grow a lot faster as n gets large. Anyway, can you guys think of any other functions that grow faster than the exponential? Might be something fun to talk about with your neighbor now, or maybe quiz your parents when you get home tonight. Anyway, thanks for watching, and we hope you had as much fun as we did.

I hope you learned as much as John did.

Hi. Thanks for listening to our lecture. The topic of exponential growth, and perhaps especially exponential growth as compared to polynomial growth, is a topic that really interested everyone on our team, which is what motivated us to make this BLOSSOMS module.

Now exponential growth is a topic that's keenly applicable to a number of different real world problems. For instance, population growth is often well modeled as an exponential function. Population growth is one of the huge long term issues facing policymakers around the world.

Another example, and one that really interested the members of our team, deals with algorithm run times. Algorithms are used to process data all over the world for many important tasks. If you have an algorithm that runs in an exponential runtime, your hands are kind of tied, you can't give it too much data, or you're simply never going to get an answer back, because exponential functions grow so rapidly.

Providing insight into just how fast exponential functions grow as compared, for instance, to polynomial functions, we think is a very useful thing that's often not covered in a typical high school or perhaps even college curriculum. Of course, as you know, you have a key role in motivating the activities and kind of tailoring it to your classroom and making it the most accessible to your students. We thought we'd just provide a few suggestions that we had thought of that might come in useful in that process.

So we think that this module should be accessible to a wide range of high school students, but there are a couple of core competencies that I think students probably wouldn't get that much out of the module if they didn't understand. For instance, being able to evaluate polynomial or exponential functions is pretty crucial, because that's a lot of what we're doing during the course this module. Understanding of scientific notation is also important, because we often present large numbers, which we come across a lot, of course, in a module about exponential growth, we're going to use scientific notation to represent those.

There's some advanced concepts that are probably not crucial to fully understand. We think that students will be able to understand the point, but students who do understand those more advanced often calculus level concepts might be able to get a little bit extra out of the module. Those include the convergence of functions, limits of functions, and mathematical induction.

So the first activity segment we present deals with an Indian queen, who is paying out rice, one for the first square, double for each additional square. And of course then, the key understanding to being able to manipulate this problem is to know the number of grains of rice that have been played down by the nth square on the board. We suggest that if students are having a tough time understanding exactly how the problem is set up, that you might start them off with a two by two, a three by three matrix instead of an eight by eight, which is a bit large, and have them actually work out for each square the number of pieces of rice that have been laid out on the board.

Perhaps, listing those payouts for n equals 1, 2, 3, 4, 5 next to the numbers 2 to the 1, 2 to the 3, 2 to the 4, 2 to the 5, might help the students understand that the general formula by the nth square is 2 to the n minus 1, which is again the key comprehension to be able to proceed and answer a question like could the queen actually pay out 64 squares of the chessboard.

The second segment deals with the problem of tearing and stacking paper on top of each other. And specifically, we ask a number of questions related to just how many times you need to do this operation of tearing paper in half and stacking it to get a stack of paper to a certain height. So the key understanding to be able to answer any question of this sort is exactly how many papers we have in our stack after the nth tear.

Students may quickly get this depending on their level of mathematical sophistication, but if they don't, we suggest just having students tear sheets of paper, put them on top, and they should be able to get the trend pretty quickly. The fact that after n tears, there are 2 to the n pieces of paper in the stack.

Once they understand that, the questions we think require a bit of insight in the students into the kind of key base question, which is just how many pieces of paper do you need in a stack to get to a certain height. We've given a piece of data, which is that a piece of paper is 0.1 millimeters thick, which is 10 to the negative fourth meters. But students should be able to figure out for themselves that they need to know the typical height of a human being and also the distance to the moon.

If the students have a way to easily-- they should know roughly how tall human beings are, about 1.75 meters tall, but if they have trouble figuring out how far it is to the moon, you could probably give them that piece of information, it's right around 384,000 kilometers away.

So once students have figured out exactly how many sheets of paper they need to reach a certain height, they're in the home stretch. But they still need to actually convert that to a number of tears to answer the problem. So of course, this is really a logarithm, the log base 2 of the number of sheets of paper that they need, and that would give them the answer, the number of tears. And students may be able to compute this with a graphing calculator.

We found that many graphing calculators don't have a log base 2 functionality, so there would need to be some sort of change of base, perhaps taking a log base 10 and then dividing that result by the log base 10 of 2. So it might actually be easier to just do the look up in the powers of 2 table to figure out the number of tears. In this case, students would just be looking for the number n, at which finally we exceed the number of sheets of paper that we need to reach a certain height. And this is how we present the material at the beginning of the next section.

So in the third and fourth sections, we introduce the concept of which function grows faster. Is it an exponential or is it a polynomial function such as a linear or a quadratic function. And when asked on a range, like on the time range of a 30-day month, this is probably best addressed by plotting the functions. We particularly chose functions that are somewhat easy for students to plot, and we think that students should be able to plot the graph of 2 to the n versus 10,000n without too much trouble.

Of course, if they have graphing calculators available to them, another option would be to plot on that domain these two functions and to use the graphing calculator's ability to intersect functions to find an exact point of intersection, which of course would correspond to the exact day at which there's a payout kind of break even point and when we start making more tootsie rolls, Swati starts making more tootsie rolls than John.

So the fifth segment is certainly the one that's most mathematically involved. We introduce the concept of in general for an arbitrary polynomial function and an arbitrary exponential function with base greater than one, which one ends up growing faster as n approaches infinity. So there are a couple of concepts here that are very intimately related to calculus. And we recognize that some of the students listening to the lecture might not have already taken a calculus course. But if so, there's kind of natural links to limits, for instance, and to the ratio test.

We also think that if there were teachers of calculus classes, who wanted to look at alternate derivations of this fact, for instance, using repeated application of L'Hopital's rule that could be a really interesting and useful exercise for students, who are in that calculus class.

Thanks for your time, and we hope you and your students enjoy exploring the world of exponential functions with us.