The Quadratic Equation: It's Hip to Be Squared
By Professor Gilbert Strang

I'm Gilbert Strang. I teach at MIT, a wonderful job, and I get to teach you today about quadratics. So let me say, what is a quadratic? There's ax squared. It's that square that makes it a quadratic. b times an x and a c, a constant. So that would be our function. If you give me an x, these a and b and c are numbers. And you'll see, I'll change those numbers to get different quadratics. But this is the whole family of quadratics. And if you give me an x, and I know a and b and c, then I can figure out y. It that's squaring that makes the word quadratic come into it.

OK So a good start is-- connect the algebra, these letters, to the graph, the picture. The geometry. So I've drawn three quadratics up there. Three particular ones. Let me tell you what the a and the b and the c are. You see that those quadratics are almost the same shape, just lifted up. So this first guy is going to be-- y is-- I think I picked x squared minus 2x there, for this one.

Now this one is up a little higher. The c is what moves it up and down. That's the easy part. It doesn't change the shape, just shifts it up. So I think that second guy is going to be x squared minus 2x. Same thing, but plus 1. So I've lifted it up by 1 to get the second one, and then the next one is lifted up by another 1. So the top guy is y is x squared minus 2x plus 2.

OK. So those are three particular quadratics. And what do I ask you about them? Let me start by asking when is y 0? I'm looking for x's that produce a 0 here. And I won't work right away with this general a and b and c. I'll work with these numbers, where a was 1, b was minus 2, and c changed from 0 up to 1 and up to 2. So I'm looking to see-- well, let me take the bottom one, the first one.

So what points am I looking for? I'm looking for-- there is one point, and there is another point. Then I'm planning to figure out-- that's, you could say, solve the quadratic. Find where it's 0. So I would like to find where this guy-- that's the bottom one-- is 0. Not too hard, but we will do it in a good way.

So I want to find where that 0, and maybe you know that one way to do it, if you have a nice example like that one, is factoring. If I can split this into-- multiply two things together, and I can-- do you know how I could factor that? An x appears both places, so I can take out the x, and what's the remaining thing? That x squared, of course, means x times x. So I need an x times an x there. And here I have minus 2 times the x. That's what I would call factoring that quadratic, because it's got that square in it.

OK. And this tells me what I could probably figure out anyway. It tells me where the 0's are. When is this 0? Well this could be 0. That's this point, x equals 0. That's that point. Or x could be 2, and then this would go away. And if x was 2, I'd have 4 minus 4. I'd certainly get 0. And that's this point. So the two roots are-- shall I introduce those words, roots? I don't know why we use the word roots. A lot of math words, who knows why? The roots are 0 and 2. Good. I have solved that first quadratic.

OK. Ready for the second guy? This one. What do we see in the picture for that one? That's the middle one. And you see it's rather special. It touches the y equals 0. It hits y equals 0. It has a root at just one place. This had a root twice. This one has a root just once. And I can find out where that is.

Well, shall I try the same thing of factoring? Shall I factor this guy? So, I see x squared minus 2x plus 1. Maybe that might jump to mind as-- it's something times itself. That's a very special one to factor. That's x minus 1 times x minus 1, what I would call a double root. That is 0 at x equal 1. So what am I saying about this guy? Double root 1 and 1. Double roots get mathematicians excited. Well, mathematicians are easily excited, you could say. And that, especially.

OK. So it's x minus 1 squared, and we see a double root. Now let me go on to the top guy. Where is that 0? Can we factor that? We have a problem, this one, because we can see from the picture that this thing never hits 0. So am I going to say no roots for the third guy? No roots. I don't like to say that.

Gauss, who was the greatest mathematician of all time, said that if we have a quadratic, it should have two roots. If we had x to the 27th starting our y, there should be 27 roots. But where are they? OK. And you might say, well, just factor and find them. Well, of course, I can see from the picture. I'm not going to find them. If I try to factor this in this way or this way, I don't succeed. What's up here? What's up with this case where the quadratic is up there?

Let me pause here, and give you a chance to think what to do, see what the problem is, and maybe think of another example. Let me suggest some other examples. A different a, b, and c. Suppose I take a to be minus 1. That will change the picture completely. With that minus sign, my quadratic is going to go downwards. It'll be like throwing a ball in the air and having it come down. It's going to go that way. And let me take, say, 4x here, and I don't know what to take as c. Let me take 9. That seems a pretty big number. I don't know if that has any roots. I don't know if you could graph it, but you could sure try. So have a try, and then I'll see you in a minute.

OK. Hi again. I'm back and still looking for those missing roots. You remember, we had this example of x squared minus 2x plus 2. So that was a particular choice, if a was one, b was minus 2, c was plus 2. And we couldn't see the roots, and our picture didn't show any. And the reason is, we're going to have to go to imaginary numbers. I like to think that root is still there, those two roots are still there. But they aren't going to be real roots.

In fact, I'm going to have to write down the formula that everybody hates. It's the formula for the roots of this thing. Roots, you remember, I set this to 0 and I look for x. And I'm going to find 2 x's, two roots, where this thing comes to 0. And I'm going to stay with algebra for a minute and write the answer in terms of a and b and c, because if I know those three numbers, then I've got the quadratic. I've got a picture. I've got the roots. Only, either I've got them or they're somehow imaginary.

But here's the formula for the two roots. OK. So the roots are, x equal. OK. There's a minus b. No problem. There is a plus or a minus. That's why I'm going to get two, because I can take the plus answer or the minus answer. And then I have a square root-- it's going to involve all of a, b, and c, in this special form-- b squared minus 4ac. And then I have to remember to divide by 2a. All right. Sorry about that. Can't be helped.

So that, you see-- a, b, and c are appearing here. Oh, well, let's just check it. Let's check it on the ones that worked. So this was the case when a was 1, b was minus 2, and c was 0, right? Can I try this formula, maybe over here? I'm hoping to get this answer, 0 and 2.

Let's see if I do. It's satisfying to think, OK, this formula works. The b is minus 2. And there's no c. The c is 0. So I'm going to put-- there's my a, b, and c. For practice, I'm going to put them into this formula and see what I get. Hopefully I get these roots.

All right. Let's do it. This is my x. Remember what we're finding. We're finding the x is where the curve hits the x-axis, hits 0. OK so minus b is minus 2. Minus minus 2 is x is 2. Minus minus 2. Plus or minus-- now comes the little dodgy part. B squared-- that's minus 2 squared, is 4-- and I divide by 2a. A was 1. You see how they all come in? 2. And what do I get then? This is from my big formula. I put in the numbers. Now the square root of 4 is 2. So I have two possibilities here. 2 minus 2, which would be 0. or 2 plus 2-- square root of 4, 2-- would be 4 divided by 2. I would get 2. And sure enough, that's what I wanted. The formula checked out. The formula checked out to give me those two roots, 0 and 2.

Can you do it once more? Well, twice more, actually. Once more. I'm thinking I'm going to get the next one, I think I'm going to get 1 and 1. Let me erase this and do the same thing for the second one, where c is now bumped up to 1. OK.

So again, I have minus b. That's minus minus 2. That's a 2. Plus or minus the square root of b squared-- is 4. And now I have minus 4-- this is coming in, now-- minus 4. a is 1. c is 1. Minus 4. That's not bad. Divided by the 2a, which is 2. OK. I plugged it in, The new a, b, c, with c equal 1, now, in there.

And what do I have here? This square root is 0. 4 minus 4. That's a very special case. So now I have 2 plus 0. And 2 minus 0. 2 both ways. And then I divide by the 2, so I got 1 twice. Just what I wanted. Just what I wanted. The double root. The double root happens when this square root is a 0. That's when you get the double root, because then the plus and the minus are both plus 0 and minus 0. No difference.

All right, third one. This is the new one, now. This is the one where we're going to find a root where our picture doesn't show one. OK. So now, what's the deal? c is bumped up to 2 now. OK. So c is 2. And I go again, last time with this formula. But now I'm going to get something I didn't know.

So minus b is a minus minus 2. That's the 2. Plus or minus the square root of b squared. That's the 4. Minus 4ac. Now let's get it right. Minus 4. a is 1 and c is now 2, so that's minus 8. Minus 8 divided by 2a, which is the 2.

What have I got now? I've got a square root of a negative number. That's what the problem was. That's why I couldn't see it in the real world, because it's not there in the real world. 4 minus 8 is minus 4, right?

So what's the square root of minus 4? What's the square root of minus 1? That's where we're going imaginary. The square root of minus 1-- there isn't any real square root of minus 1. So we invent one. i for invent. i for imaginary.

So the square root of minus 1. Can I write that great fact up here? The square root of minus 1, we're going to say, is i. i for imaginary. Now here I've got the square root of minus 4. Well, the square root of 4 is 2. It's the minus 1 that's making me imaginary. So this, for me, is 2i. 2 plus or minus 2i divided by 2.

Why 2i? Because if I square-- what's the square of 2i? Let's just check this out. What is the square of 2i? i squared is minus 1. I can put up here, i squared is minus 1 and 2i squared-- so I have the 2 twice, that's the 4. The i squared is the minus 1. It's the minus 4. That's what I wanted. That's why that square root is 2i.

Now-- oh, I get to cancel the 2's. Can I just do that? 2 divided into 2 gives me the 1. 2 divided into that 2 gives me the 1. Look what I've got. 1 plus or minus i.

Two roots, as we hoped, as Gauss was sure we would find. One plus i is a root. 1 minus i is the other root. Neither one is real, but they're both-- complex numbers, is what I should say. A complex number is a number that has a real part-- the 1-- and it has an imaginary part. And here we get an example with both, and it connects to the picture when the curve doesn't actually cross the x-axis. And that's because I only knew how to draw a real picture.

OK So that's the answer. Those are the missing two roots that we were looking for. OK I want to give you something to do in the break, and then I'll come back with a new business about quadratics, and actually, it's going to be calculus. You're in for a treat. But let's finish this formula by giving you practice. And I'm just going to change these minus 2's to minus 1's.

So I'm giving you three different pictures. The b is now going to be minus 1. I'm going to ask you to find the roots, where x squared minus x is 0, where that's 0, where that's 0, and I don't know if you're going to get an imaginary one or not. But I think you are. I think you're going to get an imaginary-- imaginary numbers are going to come up here. Real numbers are going to come up here. And I don't know what's going to happen for that middle one. OK, enjoy.

Well, welcome back. I've drawn the graph of a quadratic on the board, and there is one point on that graph that I want to find. One point jumps out, there. It's the bottom point, the minimum of the quadratic. On the left of that, the function's going down. On the right, it's going up. We want to find the point where it's holding still for a moment.

So let me choose a particular quadratic, say x squared minus 2x. So that's the graph. I guess that if that might be-- so let's run the x-axis along here and the y-axis up as usual. So this is x equals 0 there. And here is x equal 2. So previously, those were the points we looked at, the places where y was 0. Now I'm interested in the bottom, that point where the graph bottoms out.

OK. How to find it? Well, I get to tell you about calculus. Why not? We want to know the point where the slope-- you see the slope is downwards here? The graph is coming down? At this moment, the slope is 0, and then after that the slope goes plus. It's positive. So I really want to know the slope of this thing so that I can find a point where the slope is 0. Now, so the key idea of calculus is to find the slope.

What do I mean by slope? Slope stands for distance up over distance across. That's what slope means. And if I take, say, between this point and this point, the distance up over the distance across, that doesn't take into account the key fact that the slope is changing all the way. So this is the problem that had to get solved.

So what I'm telling you about is one of the most important ideas in the history of mathematics. And the idea occurred to two people, Newton and Leibniz, at more or less the same time. This is a fantastic story. So Newton, Isaac Newton, in England and Leibniz in Germany and France-- they were both extremely proud of this idea and wanted credit for it, and they weren't friends as a result. So I have to tell you the idea.

OK, so again, what's the problem? The problem is finding the slope when the slope is changing. So here's the idea. I want the slope at a typical point where that typical point is x. What I'm going to do is look a little to the left of that point, x plus h, and a little to the right of that point, x minus h and x plus h-- so this x minus h and this guy is x plus h-- and figure the slope between those two points. You see?

Shall I draw that picture a little bigger ? Here comes the changing slope. I would like to know the slope there. What's that slope at that point? But the way to do it is look next to it, look above it, and look at the straight line-- it won't go through that point. Everybody sees that? It goes through those two points, and we can find its slope, because we know that the distance across is 2h. You remember, this was x, and this distance was an h, and this distance was an h. So the distance across is 2h and I just want to find, what's the distance up there?

OK, here we go. So I have to find y at this point and y at this point, and subtract to get that height. To get-- how far did we move? OK. So let me take the x squared term first. So the distance up from the x squared would be the x plus h squared minus this height, which is the x minus h squared, over 2h. I'm just focusing on the x squared first because that's the part where the slope is changing. The slope here, as we'll see, will just be a fixed number, minus 2. Because that's a linear term. That's not changing, but the slope here is changing.

So this is that one minus this one, and I have to simplify it. But now we get to use algebra. So if I simplify this. This is-- everybody knows, x squared, 2hx, and h squared, right? That's the square there. And then I subtract x minus h squared. That's x squared minus 2hx plus h squared, right? X squared here, minus h squared is that, and then the cross term gives me that, and I'm dividing by 2h.

What am I getting? I cancel. Look, x squareds cancel each other. h squareds cancel each other. I'm left with 2hx minus minus 2hx. So I'm left with 4hx divided by this 2h, the distance across, and what do I get? I get 2x. 4 divided by 2 is the 2. h's cancel. And x. That's the slope of the y equal x squared. If you want to know the slope of a parabola, of a quadratic y equal x squared, be one of the, I would say, maybe, one million people in the world-- know that answer, that the slope is 2 times x. And you're now in that group.

OK. Now I'm looking for the slope of y, so I have to remember this part. If I do the same here, it'll be minus 2. This part is a steady, no curve, here. The curve is coming from there. So the slope of y is 2x minus 2. So this was the part from the x squared, and now the final answer is slope equals 2x minus 2. That's the answer that calculus gives.

And now I'm ready to use it to locate that bottom point. So the key step in using calculus, in using the slope, is to set it to 0. Because this is the point where the slope was negative here, 0 for one instant there, and then positive again.

And when does 2x minus 2 give 0? So I'm looking for the slope to be 0. I'll erase Newton-- no offense. The slope is 2x minus 2. And I want that to be 0. I'll erase Leibniz, to be fair. Where that 0 is, x equal 1. x equal 1 is that value of x. That's the x, x equal 1. Now I could say, what is the bottom value? The bottom value at x equal 1, y is 1 minus 2-- it's down one. So y equals minus 1 is a minimum of that parabola, that graph.

Let me give you a question during the break to answer. It'll give you a chance to use this idea, the Newton-Leibniz idea. Suppose my graph, my quadratic, was going downwards instead of upwards. Let me give you an example. So here's a question for you. Look at that process that we did. y at this point minus y at that point. And do the same for a different quadratic. So I'll just erase what we did there and show you the quadratic I'd like you to tackle.

Here you go. This is an [? L1 ?] that goes this way. What will be the equation for that? Y will be minus x squared, because as x increases-- big to the left, big to the right-- we need a minus sign to come downwards. Plus, let me say, 50x. So my question for you is what's the slope at a typical point x?

And then, of course, we're going to look at the special point x, which will be that guy at the top. That'll be a maximum, now, but again, it's going to be a slope 0. Slope positive-- coming up, climbing the hill, top of the mountain. Slope negative-- coming down. The top is identified by slope 0.

OK. There you go. Minus x squared plus 50x. What's the slope?

OK. Last time we found the slope and I want to use that. I want to show you a typical calculus problem. We're really moving here to get straight into calculus. But what I always hope is to say, this isn't too difficult. You can do it.

OK. Here's my question. Practical question. Suppose I have 100 meters of fencing. 100 meters of fence. My question is, what is the largest rectangle? I'm a farmer. I want to fence in my cattle, give them as much area as possible for feeding, and keep them under control. I've got that much fence. What shape rectangle should I construct for them?

OK. So what do I mean by largest? I mean the greatest area. So let me do some examples. Here's an example. I could have a long narrow region, say only 1 meter across, 49 up-- that uses 50. Over here, 49 and 1, because it's a rectangle. We'd use the other 50. The area of that would be the base, 1, times the height, 49. Just 49, in fact. That's probably not the best rectangle.

Let me change to a second choice. Let me use 2. That one was too narrow. Let's make it a little wider. So it won't have as much height. What have I got for height, now? If that's 2, I probably want 48 now. And this will be the 2, and this will be the 48. And what's the area now? This one had an area of 49, and now the area here is 2 times 48-- 96. So that's better.

But is it the best? So let me-- now I have to use algebra. The idea of algebra is, use a letter x to denote any width. If we only have arithmetic, we're playing with that number, that number, the next number. Algebra says let this distance be x and figure out the rest. Well, x up here. What do I have here?

Let's see. That was 48 when this was 2, 49 when it was 1. I think we're going to use 50 at this part. So that's 50 minus x, right? And this is also 50 minus x. And now I've got a rectangle it uses a total of 50 there total of 50-- it uses my 100 meters. So my area is the base times the height, 50 minus x. Or if I multiply that out, I see the minus x squared that we-- and the 50x. And I want to make that as large as possible. I'm looking for the top of this parabola, the maximum of this quadratic. The very question that you were thinking about in the break.

So did you figure the slope of that? Let me just put it up here. The slope-- because you remember, if we want the maximum area, we want the slope to be 0 at the maximum-- If I graph this thing, it is one of these parabolas that goes up and down, and I'm looking for that point at the top. That's the max. OK, so the slope, by Newton, Leibniz, and you, is with a minus sign-- I hope you saw that it was minus 2x now. And the slope of the 50x is just 50. That's just the easy part. And at this point, that's the point where slope is 0.

So there is our equation for the best x. You see, algebra paid off, because we considered all x's. Calculus paid off because we found the slope. We solve that equation we get minus 2 x plus 50. That means 2x will have to be 50. x will be 25. is best. And what do we see, what's the picture looks like when x is 25? If that's 25 then this is also 25. We're getting a square now. The square here is 25 each way. 25, 25. And the area? What's the area? Can you square 25 in your head? Sure. 625. I've always been hoping to be asked to square a large number. 25 is not very large, but it'll do. 625 is the area. And we think that that's the maximum. That's what the farmer should do, provided he wants a rectangle.

Now I have a final question for you that I'm not going to answer. Can you find a shape? It won't be a rectangle-- we've got the best one. Can you find a shape, different from a rectangle, where you use the same 100 meters of fencing but you enclose more than 625 square meters? That's the question. I hope you get it.

Hi I'm Gilbert Strang and thanks for teaching math. We enjoy it. Can I tell you a little bit about what to expect in this video? So this is more than the other BLOSSOMS video. It's closely connected to what happens in class, what we all actually teach. And it's about the special quadratics, ax squared plus bx plus c. And I want to try to make those familiar, partly by drawing pictures of the graph and partly by dealing with the places where y is 0, the roots of the quadratic.

And I'll have a shot, in the second part of this video, of taking that miserable formula and using it for some particular examples. So, making it familiar. And I'll even-- if it's OK with you-- I'll even take an example in which this guy turns out to be negative, and then its square root is an imaginary number. And you get the fun of asking the class to imagine a number that they've never seen before.

So part one will be the graphs. Part two will be using the formula. Now, for part three, I figure out the slope of this curve. The slope of a curve, of course, that's a calculus question.

So I'm really taking a leap into a few minutes of calculus. I don't think that's bad. I don't think it's always required to prepare everything with definitions first. I'd rather ask, what's the question? What's the idea about it? The slope here is the central idea. The slope is changing. So we all know that Newton and Leibniz figured out a way to deal with that possibility of changing slope.

And then in the fourth part, I'll do an example, a classical example, where you're looking for the maximum. Here is the minimum. If it was going the other way, it would be the maximum. The place where the slope is 0. So you get to use calculus.

So one is pure algebra. The second is algebra with a complex number coming in. The third is a touch of calculus, the slope. And the fourth is an application of slope. An application of calculus.

The students could come back to this. They could see it now and come back to it later. And I hope they find it useful. Thank you.