THE ART OF APPROXIMATION
BY STEPHEN HOU

Hello everybody! Welcome to the "Art of Approximation in Science and Engineering -- How to Whip Out Answers Quickly". My name is Stephen Hou and I am a graduate student in Electrical Engineering and Computer Science at MIT. Today we'll be exploring some concepts in ways that you may not have seen in your math and science classes. Scientific calculations can often be long and tedious. In school many textbooks and teachers often emphasize getting exactly the right answer, but many times getting an approximate answer in a much shorter period of time may well be worth the time saved. In this video module we'll be exploring techniques for making quick, back of the envelope approximations, that are not only surprisingly accurate but also illuminating for building intuition and understanding science.

So let's look at a quick example. Think about how to calculate 29 × 31. Now try to do this without using the standard rules of multiplication that you learned in school. Talk to your neighbor for a few minutes and we'll come back. See you!

Welcome back. Let's take a look at the problem 29 × 31. Well, the first approximation that we can make is that 29 and 31 are both equal to approximately 30. So 29 × 31 (NOTE: He actually says 29 × 21 instead of 29 × 31.) should be close to 30 × 30, which is a much easier calculation to make. And that is 900.

Now let's think about tweaking 30 × 30 so that it becomes 29 × 31. One way to think about multiplication is we can think of it as the calculation of the area of rectangle whose sides are equal to the numbers that we're multiplying. So a 30 × 30 rectangle, which is a square, is represented as so. We have a square of sides of length 30. How do we tweak this square to get a rectangle of sides 29 × 31? Well, we can remove a strip from the bottom of width one so that this side has length 29 and we can add a strip to the right side of width one and now the rectangle outlined in the dotted line is a rectangle of sides 29 × 31.

Let's calculate the area. Well, the area of the original square is 900. We subtracted off this strip which is 1 × 30 and therefore it has an area of 30. We've added the area of this strip and since it's 1 × 29, we add 29 square units and this gives us 899, which is indeed the answer to the question 29 × 31.

Another approach we can use is to use algebra. Now 30^2 is a squared number so one thing we learn in algebra is the difference of squares. $A^2 - B^2$ is $(A + B) \times (A - B)$. We notice that 30 not only is close to both 29 and 31, but both numbers are actually off from 30 by the same amount. And that fact is what is represented exactly here. So if we let A be 30, and B be 1, then we do get 31 × 29. So following this identity, A^2 is 30^2. B^2 is 1^2 and therefore we get 900 – 1, which again is 899.

So to summarize, in this problem we looked at arithmetic calculation, 29 × 31. But instead of working it out the way we usually do in arithmetic, we looked at inspiration from both geometry and algebra.

Now let's look at a different problem. Think about the fraction 3 ÷ 17. Now without using long division, talk over this problem with your neighbor and think about how to calculate this as a percentage. I'll see you in a few minutes.

Welcome back. Let's think about how you thought about the problem 3 ÷ 17. Well, the first thing that we know is that this is a fraction that's between 0 and 1. And since 3 is less than half of 17, which is 8.5, we know that this is a fraction that's somewhere between 0% and 50%. OK. Let's see if we can refine those bounds a little bit. Part of what's making this problem a little bit difficult is the denominator, 17, which is an unusual number to work with. Let's think of some denominators that are close to 17 but are much easier to work with. So how about 20? What if we look at the fraction 3/20? How does that compare to 3/17? Well, a larger denominator makes the number smaller, and since 20 is larger than 17, 3/20 is less than 3/17.

Now how about on the other side? If we reduce 17 even more we can get a denominator of 15, which in this case is also easy to deal with. What is 3 ÷ 20? Well, we know that 1/20 is 5%, so 3/20 is 15%. And how about 3/15? Well, 3/15 we can simplify to 1/5 by dividing the top and bottom by 3. And we know that 1 ÷ 5 is 20%. Now we have much tighter bounds. So we know that 3/17 is somewhere between 15% and 20%. Can we do even better? Well, the best denominators to work with are those that are powers of ten. So let's try to make 17 a power of ten. One way in which we can attempt to do that is what if we multiply by a unit fraction *X*/*X* such that we get denominator that's approximately equal to a power of ten. Now the next power of ten that's above 17 is 100. So 17 times what number is approximately equal to 100? Well, we know that 16 × 6… Well, we know that 100 ÷ 6 is 16 point something. So let's try 6 ÷ 6. So we multiply 3/17 by 6/6. We will get 17 × 6 = 102. And 102 is very close to 100. So 18/102 is approximately equal to 18/100. So we know that 3/17 should be approximately equal to 18%. So this is our more refined answer.

But can we do even better than this? Well, if we wanted to make 102 100, one way to think about that is to reduce 102 by approximately 2%. So if we can scale the denominator down by 2% then we want to scale the enumerator down by 2% as well. So what is 2% of 18? 2% of 18: we multiply 18 × 2 = 36. And since we're taking a percentage, then we move the decimal point by two units and we get 0.36. So if we reduce 18% by 0.36% we will get 17.64%. So going back we can say that 3/17 is approximately 17.64%. And if you use your calculator you can see that 3/17 in fact is 17.64 and then a few other digits as well. So just by you doing some quick approximations we were actually able to get 3/17 to four decimal places.

Now for the next example let's look at one from the physical world. So let's turn to the computer screen for a second. Now on the computer screen what we have is a simulation of a pendulum. So a pendulum is a mass attached to a string, and the other end of the string is attached to the ceiling. And once the pendulum is released from some initial angle, we see that gravity just causes it to oscillate back and forth. On this particular demo we can see that the path of the pendulum is outlined in red and that's the elongation that's shown at the bottom that keeps on changing in meters. The graph that you see that looks like a sine or cosine wave is a graph of the horizontal position as a function of time. So in this case the horizontal position is in meters and time is in seconds.

Now below the pendulum we see that we have the timer running -- so far it's been running for 45 seconds and counting. And below that we see the oscillation period. So the oscillation period and that's the focus of this particular example, is the amount of time it takes for the pendulum to make one round trip. So from one particular place in which the pendulum is released, we want to time how long it takes for it to return to that position. So let's see. We have a hill coming up, so now. And then we wait a little bit and there, the pendulum just returned to that point. And we felt that was approximately 4.5 seconds.

Now on the right hand side, in the green region, we see that we can actually alter a few parameters of this pendulum. So let me just stop this real quick. We can change the length of the pendulum. So right now the default is set to five meters. We can change the gravitational acceleration which is in meters per second squared. And right now that's set to 9.81 which is the gravitational acceleration on the surface of the earth. We can change the mass at the end of the pendulum. In this case it's 5 kg. And we can also change the initial amplitude which is the angle at which the pendulum is released. Now my question is if we know the length, the gravitational acceleration, the mass and the initial amplitude, how do we calculate the oscillation period of such a pendulum? So I know this is a relatively difficult question so I'll give you a few minutes to think about it with your neighbor and then we'll come back to it in a second. See you guys soon.

Welcome back everybody! Let's talk about that pendulum problem you just thought about. Let's turn to the computer. Before the length was set to five meters and saw the oscillation period was 4.49 seconds. Let's see what happens if we reduce the length to just one meter. So if we go to the pendulum we see that now it's extremely short and it bounces around very, very quickly. And according to the demo it says that the oscillation period is now 2.01 seconds. And this is something that makes sense from common sense. When the pendulum is a lot shorter, we would expect it to bounce back and forth much faster.

Now let's pause the demo, reset that, and this time let's make the length much longer. So what would happen if we made the pendulum ten meters long? So start the pendulum again. And now we see that it bounces back and forth very slowly. And according to the demo the oscillation period is 6.34 seconds, which is longer than what it was when the length was five meters. OK. So now we can safely say that the longer the length of the pendulum is, the longer the oscillation period. Now let's see what happens when we change some of other parameters. So let's change the length back to the default which was five.

Let's take a look at gravitational acceleration. On earth that's 9.81 meters per second squared. Now say we went to an asteroid or something where the gravitational acceleration was only one meter per second squared. So the gravitational field is much weaker. Now let's see what happens to the pendulum. We see that now it goes very slowly. We haven't even reached a full period yet, it hasn't swung back to where it was. It's about to soon. OK, let's wait a few seconds, and now it has. And the oscillation period in this case is 14 seconds. And this is something that we would expect as well. The pendulum swings because of gravity, so the weaker the gravitational field, the slower we would expect the objects to move.

So just like before, we'll pick a number that's higher than the default, so let's pick 20 meters per second squared. Now if we start that we see that the pendulum now swings relatively quickly. So according to the demo, the oscillation period is now 3.14 seconds. Let me pause this and return it back to the default. So now we saw that the oscillation period depends inversely on the gravitational acceleration. So as the gravitational field increases, the oscillation period will decrease.

OK. Let's take a look at the mass. Before it was at 5 kg. Let's see what happens if we use a much heavier object at 10 kg. And we'll start the demo. And we see it looks very much how it was before. And the oscillation period is 4.49 seconds, which is what it was. So we see that mass apparently does not have any effect on the oscillation period. Just to confirm that a little bit, let me reset the demo. Let's see what happens if we change the mass to 1 kg. So this is something that's much lighter than the default. So let's start the demo. And again, it looks exactly how it did before. And the oscillation period remains at 4.49 seconds. So now we can guess that the mass actually has no effect on the oscillation period. Let's pause this demo and reset that and we'll set the mass back to the default which was five. Actually that doesn't matter.

Now let's take a look at the amplitude. The initial amplitude I set by the program is twenty degrees. Now let's see what happens if we start from a much smaller angle, let's say two degrees. And we'll start the demo now. So you can see that the pendulum does not actually oscillate very much, but it still oscillates nevertheless. And if we take a look, actually the oscillation period is still the same, 4.49 seconds. Now you might think this is surprising because since we started at a smaller angle then doesn't the pendulum have a shorter distance to go? That's true, but it also has less distance to accelerate as well. And in fact those two effects almost nearly cancel each other out. So for relatively small amplitudes the oscillation period actually remains independent of that. And I should add the caveat that of course if the amplitude is much, much bigger, there is some effect. But we'll ignore that for the purposes of this example.

So let's reset and just to check, let's try a much larger amplitude. Originally it was ten degrees, let's try twenty degrees. So start the demo again and you can see that the pendulum swings much faster, but also swings through much larger region. So again the oscillation period is unchanged, 4.49 seconds. The reason for that being of course even though it has more distance to cover, it also has more distance to accelerate as well.

OK. So just to summarize, we've found that length has a positive effect on the oscillation period. Gravitational acceleration has a negative effect on the oscillation period. And mass and amplitude do not seem to have any effect at all on the oscillation period. Now let's stop this demo and go to the board and think this through a little bit.

We know that length, which has units of meters depends positively, the oscillation period depends positively on the length. As length increases the oscillation period increases as well. The gravitational acceleration *g*, which has units of meters per second squared, we know that the oscillation period depends negatively on it. So as *g* increases, the oscillation period will decrease. And we know that mass and the initial amplitude do not affect the oscillation period at all. Now only knowing the fact that length is in meters and *g* is in meters per second squared, think of a way to combine these variables in an equation that will produce the oscillation period which is in seconds. So think about this for few minutes with your neighbor and we'll come back. I'll see you guys soon.

All right. Welcome back. Let's see how we made with this particular problem. So we were given the length of the pendulum which has units of meters and the gravitational acceleration which has units of meters per second squared. And we want to find an expression for the oscillation period which has units of seconds. So we have to get rid of the meters somehow. And since both length and gravitational acceleration have meters in the numerators, if we divide the two, the meters will go away. So let's look at length divided by gravitational acceleration. $L \div g$. So this has units of meters/meters per second squared. So as we wanted to, the meters will cancel away and we're left with second squared. If we divide the other way, $g \div L$, that has units of m/sec^2/m, which is inverse second squared. Now we want to end up with an expression that is only in seconds, not seconds squared. So how can we obtain that? Let's take a square root. This is in per seconds squared, so we take the square root and we get 1/seconds which is not what we want. This is in seconds squared, so we take the square root. We do in fact get seconds. So we think that the oscillation period T should be equal to square root of L/g. Now in fact the correct answer is 2π times the square root of L/g. And in fact you should pretty impressed by this, that we were able to get an expression for the oscillation period without using any math or physics and being only off by a constant factor of 2π.

Now this is a technique called dimensional analysis where we looked at the dimensions of the variables of interest and we combined them in ways that will produce an expression of the correct units for the quantity that we want to find.

So let me give you another problem. And it's stated very simply as this. How close can you get to a black hole without being sucked in? So as a hint to get you started, I'll give you these parameters. I'll give you the mass of the black hole which is in kg. I'll give you the speed of light, *c*, which is in meters/second. And I'll give you the gravitational constant *G*.

And let's find the units for *G*. *G* comes in in the following equation. The force due to gravity between any two objects is *G* times the mass of one object, times the mass of the other object, divided by the distance squared. Distance between them squared. Now we know that force is equal to mass times acceleration. So the unit for force, which is newtons, is equal to mass, kg, times acceleration, which is meters per second squared. And now we have *G*. The two masses are both in kg and the distance is in meters, so we have meters squared. Now if we simplify this expression, then we know that the units for *G* are, well the kg will cancel on both sides, so we're left with that. We can move the meters to this side, so we're left with units of *G* are meters cubed/kg/second squared. So let's write that over here. So now we have a well formed problem. Given those three variables, the mass of a black hole, the speed of light, and the gravitational constant *G*, using the same technique that we discussed earlier, combine these in such a way so that we get a variable that only has units of meters. So talk this over with your neighbor and we'll come back to it in a few minutes. I'll see you guys soon.

Welcome back! So we're looking at the question how close can you get to a black hole without being sucked in. And the hint that I gave you was using these three parameters, the mass of the black hole which has units of kg, the speed of light *c* which has units of meters per second, and the universal gravitational constant big *G* which has units of meters cubed divided by kg second squared. Now in order to get a distance, which is in meters, we have to get rid of kg and seconds somehow. So looking at this how do we get rid of kg? Well, two of these parameters have kg, mass and *G*. One of them has kg in the numerator and the other one has kg in the denominator. So let's multiply them together. $G \times M$. Now *G* has units of meters cubed/kg seconds squared. And *M* has units of kg. So the kg will go away and we are left with meters cubed/second squared. All right.

Now we haven't used the speed of light yet. The speed of light has units of meters per second and over here we have meters cubed/second squared. So remember in the end we want to get an expression that only has units of meters. So we have to get rid of seconds completely. So how do we get rid of the seconds squared that exists in $G \times M$? One way that we can get a seconds squared is to square c. So if we squared the speed of light, we'll have c^2 which has units of meter squared/seconds squared. So now we have two expressions that both have seconds squared as part of their units. We have GM which is meters cubed per seconds squared. And we have c^2 which is meters squared per second squared. Since both of them have seconds squared in the denominator, we can't multiply them together to get rid of seconds squared, we have to divide them. So let's take $GM \div c^2$. GM has units of meters cubed per second squared. The speed of light has units of meters squared per second squared. So the seconds squares will cancel with each other. $M^3 \div M^2$ gives us meters. Therefore, let's guess that this would be the expression that we want. In fact the correct answer is two times GM/c^2. Now this is even more impressive than the pendulum problem. In order to solve this problem using physics you have to know general relativity, but in fact we were able to get approximately the right answer and we're off only by a factor of two. So as an aside, this expression is what's called a Schwarzschild radius and in fact it is the distance away from a black hole such that the escape velocity is the speed of light. And therefore if you're any closer than that to a black hole, you would have to travel faster than the speed of light, which is not possible, in order to escape it.

So this concludes the module "The Art of Approximation in Science and Engineering -- How to Whip Out Answers Quickly". We started by looking at some arithmetic problems and interestingly we turned to algebra and geometry to think about how to solve those. We also looked at some examples from the physical world as well. But we didn't use any physics equations at all. We didn't even use concepts like force or momentum even, but what we did do is we looked at dimensional analysis and used a little bit of mathematical intuition to figure out how to solve those problems. I hope that you use these techniques in your further math and science studies. I had a lot of fun making this module for you today and I hope you enjoyed listening to it as well. Thank you very much.

Greetings fellow instructor! Thank you for using my module today, "The Art of Approximation in Science and Engineering". So again my name is Stephen Hou and I'm a graduate student in Electrical Engineering and Computer Science here at MIT. The reason why I decided to do this particular module actually came about through a personal experience of mine. So I've always been interested in math and science. Like many students here I've taken math, physics, chemistry and biology in high school and continued on in college. One of the things I noticed is that in high school a lot of the emphasis really is on getting the right answer. When I got to college in fact it was even more the case. I double majored in electrical engineering and physics and a lot of it was not focused on intuition. That was something that the student was expected to develop on their own. Our homework, our exams, the lectures, the recitations, discussions with the TA, and most of the time that we spent with our instructors and with our fellow students was really simply just focused on getting the right answer. When I got to graduate school, that's when the flavor changed a lot. Of course it is important to get the right answer, but then you have to be able to understand the material much more deeply in order to have any real handle on what's going on.

So a few years ago I took a course, just a short one month course, taught by an instructor from the University of Cambridge. His name is Sanjoy Mahajan. And he called it actually “The Art of Approximation.” And he in fact worked through a lot of the examples that I talked through today. And he geared his class towards everybody at MIT. It could be a freshman who is just starting to take calculus, or it can even be a professor who has already taught many courses here at MIT already. And everybody could have gotten something out of the class. But the most important thing is that it forces… he asked the students to really think through about the problems that they were doing, and to really think, “Do I really need to put in hours and hours or minutes and minutes of calculation in order to get the right answer? Or is there a much easier method using some basic principles that gets me to an approximate answer first, so I have an idea about what I’m doing. And then if I have the time I can figure out whether it’s worth my time and effort to calculate further.”

Later on I found this course so useful in my studies that as a teaching assistant for electrical engineering courses for undergraduates, I decided to incorporate a lot of these concepts into my own personal teaching as well. And I’m glad that the Blossoms project exists so that I’m able to show these concepts to students around the world as well.

With the particular module I gave today I think that it’s appropriate for students who are already familiar with algebra. And even though I did cover some physics concepts such as the speed of light, gravitational constant, it’s probably not necessary for the students to have taken a first year course in physics. In fact, I think it would be better if they have not yet taken physics. One of the things I’ve found is that when students have been—actually anybody for that matter—has been formally trained in something, and when you ask them to solve a problem they immediately turn to the skills that they already have. But when you ask students or people who have not yet been exposed to something and you ask them a question, it actually forces them to be a little bit more creative. So I would not hesitate to use this module with students who have or have not yet taken physics, it really doesn’t matter. As long as they know what mass is, what a kilogram is, they know what a meter is, they know what a second is, that should really be enough for them to appreciate what was discussed in this particular module.

Now as for the breaks in this lecture, one of my favorite models for teaching is to allow students to interact with each other as much as possible. The model that I chose to do here is that I pose a question and I give students a few minutes to think about it over with a partner. I imagine that you’re probably teaching this in a classroom of a few dozen students, and for that it’s probably most appropriate to ask the students to form groups of two, three or four and discuss the problem informally amongst themselves. Although it is true that these problems can certainly be done either individually or with the whole class together, logistically it’s probably easier if you restrict these groups to sizes of two, three or four.

I don’t imagine that students would be talking about these problems for more than say five minutes at a time. I would recommend that you certainly give them enough time to digest what’s going on, to form their own thoughts. But as soon as you start to see groups either stalling, they can’t come up with any new ideas, or if they’ve already come to the correct or incorrect answer, then that might be an appropriate time to restart the video and allow them to continue on.

As is always the case with any large classroom, you might find that you’ll have students of different abilities or some who are more familiar with the concepts or get the concepts a lot faster and will finish earlier. In cases like that, you may want to mix things around a little bit. Maybe pause the video for a longer period of time, give students a chance to show other students perhaps on the board their thoughts. This video module is really designed to fit within a one hour period with brief breaks in between. But of course if you have more time, then you'd be able to be a lot more creative with how you use those particular breaks.

Well I hope you enjoy the use of this particular module. I had a lot of fun creating it. And I hope there are a lot of students in the future who will benefit from it. So thank you very much and I hope I will meet you one day.

END OF MODULE