GUESS THE LAST BALL

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Hello, my name is Fakhar Lohdi and I am a professor of computer science at the National University of Computers and Emerging Sciences Pakistan. Today we are going to have some fun and we'll play a few games. And hopefully we'll learn something through the process. The first game that I'm going to play with you is a very simple game. So I will introduce the game to you and I'll show you the rules of the game and then I'll ask a few questions and hopefully, towards the end, we will be able to answer those questions. So let me introduce the game to you.

In this game I have two types of balls. I have solid balls and I have striped balls, _____ pool or billiard balls that we have. And the game is very simple. I will put a few balls in the basket. I'll take a few balls, some striped and some solid, and put them in the basket. Now I will start playing the game. So in this game what I do is I take two balls without actually looking at the ball from the basket and I then look at the type of these balls. In this particular case both of these are striped. So the rule is very simple. The rule says that if the balls are of the same type, then I put one solid ball back into the basket. So I take these two balls that I have pulled back from the basket. I will put these balls on the side. I've taken a solid ball from here and I'll put this solid ball back into the basket. Then I'll keep on playing the game. Again, I take two balls out. This time I have two solid balls. Once again the same rule is applied. That is if the balls are of the same type, that is if balls are both solid or both striped, then I will put one solid ball back into the basket. I play again, so I take two balls. This time I have a solid ball and a striped ball. So if the balls are of different types, not of the same type, then I will put the striped ball back into the basket. So these are the only two rules that I am using to play this game. That is if the balls are of the same type, I will put a solid ball back into the basket. And if they are of different types, then I'll put a striped ball back into the basket. And I will keep on playing this game until I am left with only one ball in the basket. So I started with a few balls as I have taken these two balls back, one striped, one solid. So I'm putting the striped back. Then I take the two balls again, solid and striped, so I put the striped back. So every time what I'm doing is I'm taking two balls out and I'm putting one back into the basket. And that means every time the number of balls that I have in the basket, they are reduced by one. So that means if I started with 10 balls, after 9 iterations I'll be left with only one ball and that is the last ball.

So the question really is given the number of solid balls and striped balls in the basket, can we tell the type of the last ball? So let me play a game again with you and then show you how the game is to be played and then we will ask this question again and I'll let you try to figure out the answer. Let me play the game. So I take let's say four solid balls. I put them in the basket. I take four striped balls and I put them in the basket. So I have 8 balls in total, four solid and four striped. Now let me play this game with you. So I take two balls out. In this particular case I have a solid ball and I have a striped ball. So the rule says if the balls are of different type, then I will put the striped ball back and I'll put the solid ball on the side. Then I go inside again and I take two balls again. In this particular case both of these are of the same type, both striped. So I'll put them on the side and I'll take a solid ball from this pile of balls that I have and I'll put the solid ball back into the bucket. Now I go inside again. Both of these are different types, so I will put the striped ball back and the solid ball comes here. I go back. There are of different types

so I will put the striped ball in the basket and take the solid ball away. Then I go back once again. I have different types so I put the striped ball back and put the solid one here. Then I go inside again. Now I have two of the same type, both striped. So I will put them aside and I'll take a solid ball and put it back. And then I am left with only two balls in the basket, so I take these two balls. Now both of these are solid, so what I do is I'll put one solid ball back and I'm left with only one ball and that is the color of the last one. So the general question is, given X number of solid balls and Y number of striped balls, that is a given. We know the number of balls that are there in the basket. Can we tell the color of the last ball in the basket when we play the game. So that is the question and I'll let you figure that out by playing a few games and observing if you can see something.

Welcome back. I hope you have played a few games. Have you been able to figure out the answer? Have you been able to guess the right answer, the type of the last ball? Does it look like a random guess? Does it seem like it depends upon the sequence of moves? Or is it something which is independent of the sequence of moves? Let us try to understand. Let us try to play a few games and let us try to observe this thing systematically.

So let us once again go back and start with a one ball game. That is we have only one ball in the basket. So if we have only one ball, then there are only two possibilities. That is I can have a striped ball in the basket. If there is only one striped ball in the basket, I pull that and that is the type of that ball. If it was a solid ball, obviously the answer is trivial. That is the type of that last ball because I started with only one ball and hence that was the type of that ball.

Now we can play a two ball game. With two balls I have three different options. I can have two solid balls. So these two solid balls are in the basket and that is what we started with so I pull these two solid balls back and according to the rules I will put one solid ball back into the basket and that is the type of the ball that I will have in the basket. So if I started with two solid balls, I will be left with only one solid ball and that will be the type of the ball in the basket. Now the second option is I start with two striped balls. So if I start with two striped balls I pull two striped balls. The rule says that I will put one solid ball back and I take the solid ball and put it here and since I started with two balls, that is the last ball and the type of the last ball is a solid ball. So the second option was two striped balls and I ended up with one solid ball. And the third possibility is that I started with two different types of balls, one striped and one solid and I put them there and then I pull them back. I will put the striped ball back into the basket and take the solid ball here. And that will be the color of the last ball or type of the last ball that is the striped ball that I have. So the third option was solid and striped and I was left with a striped ball. So these were the three different two ball games and in that case if I started with balls of the same type, I ended up with a solid ball. If I started with balls of different types, then I ended up with a striped ball. So I'll note these things now.

Now I play a three ball. Now in this particular case I have four different possibilities. All solid balls. Two solid balls and one striped ball. One solid ball and two striped balls and all striped balls. So these are the four possibilities that I have. So let us exercise these possibilities and let us see what happens. So I start with three solid balls in the basket. I put them here. I take two solid balls. According to the rule I will put one back into the basket. Then I have two balls, both of these are solid. So I take both of these out and I'll put one solid ball back and the last ball

that I have is a solid ball. Now the second option is I start with two solid balls and one striped ball. I take two balls out, one striped one solid. So I'll put the striped ball back into the basket. Then I'll take the remaining two balls, one striped, one solid and I'll put the striped ball back into the basket and I will be left with one striped ball. So if I started with two solid and one striped, I will have the striped ball back. Next is I have one solid and two striped. I put them there. I take two balls out. These are one solid, one striped, so I'll put the striped back into the basket. The next is I will take these two balls, both of these are striped so I'll put a solid ball back in the basket and that is the type of the last ball. So I started with one solid and two striped and I ended up with a sold ball there.

Then the last one is I have three striped balls. So with these three striped balls I will pull two from the basket. Both of these are of the same type so I'll put a solid ball back into the basket. Then I have a solid and a striped so I will put the striped ball back into the basket. I take the solid ball way and then the type of the last ball is the striped one. So that is what I am left with.

And I make a note of all these four options that I have. That is when I started with three solid balls, the type of the last ball was solid. When I started with two solid, one striped, the type of the last ball was striped. When I started with one solid and two striped, the type of the last all was solid. And when I started with three striped balls, the type of the last ball was striped. So do you see a pattern here? Do you need to play some more games to observe the pattern? So please play a few games, just go systematically and see if you can identify a pattern and then use that pattern to tell the color of the last ball or the type of the last ball that we have.

Welcome back! I hope you have been playing the game and you will have figured out the right answer. And hopefully your teacher will have helped you in coming to the right answer. Many of you at least would have been able to do that and those of you who have been able to figure that out, I guess the answer is, your answer would be that it is the answer lies in the number of striped balls. That is if you started with an even number of striped balls, you will end up with a solid ball. And if we started with an odd number of striped balls, we end up with a striped ball. So the number of striped balls that determined of this game. And actually the number of solid balls does not come into this play. We have solid balls but you know the answer does not depend upon the number of solid balls.

Having concluded that, now the next question is how sure are you about this outcome? Can we say that if we had 1,000 balls in the basket, we would still have the same answer? We have played games with a few balls, 5, 10, 20 balls maybe up to a maximum but you know is that the _____ answer? In mathematics what we do is these examples of doing anything, they help us in building our confidence in some conclusions that we have, some observations that we see and based upon those observations we draw conclusions. And the more we observe the better confidence that we have, the more confidence we have in our conclusion. But that does not prove anything. A very simple example is that I can make a statement that all integers are less than 100. Obviously this is an incorrect statement but I can give you infinite examples that fulfill this statement. As you all know, all negative numbers are less than 100 so I can keep on citing negative numbers and I'll show you, "See this number is less than 100. This number is less than 100. And hence my statement is true." So I can make a statement, I can cite examples but that statement may still be untrue. So in order to prove something we have to build a methodical

model and use some kind of logic to be sure that the statement that we make is true or false or whatever the answer is.

So what kind of model can we make here in this particular game? So in this particular case we can have a very simple model. All we need to do is translate the rules of the game in mathematics. And what are the rules? Very simple rules. We have two basic rules, that is when we have balls of the same type we will put a solid ball back into the basket. And when we have balls of different types we will put a striped ball back into the basket. So let us try to understand, let us try to write this rule. So this is a function of the number of striped balls and the number of solid balls in the basket. So our rule is when I take two balls out, I have three different possibilities. I take two solid balls out, so the number of solid balls are reduced by two by taking these two out. And then the rule says that I put one solid ball back into the basket so that means when I took two solid balls from the basket, I will end up with one less solid ball in the basket and the number of striped balls will not change. So the number of solid balls will be the starting number solid balls minus one and the number of striped balls will remain the same. In the second case I have two striped balls. So I take these two striped balls. So I reduced the number of striped balls by two in the basket and I put one solid ball back into the basket. That means the number of solid balls will be implemented by one and the striped balls will be decremented by two in this particular case. And the third possibility is that I have a solid ball and I have a striped ball. And in that particular case that means by taking these two out I have reduced the number of solid balls by one and I have reduced the number of striped balls by one. And then when I put a striped ball back, that means the number of striped balls will be incremented by one so it will remain the same that we started with. And the number of these solid balls, they are decremented by one. So these are the three rules. These are the three things that we can do and I have translated these into simple mathematical formula.

Now as you can see and observe these rules, we see that the number of striped balls are either reduced by two or they remain the same. And on the other hand, the number of solid balls are either incremented by one or they are decremented by one, depending upon the case that we are dealing with. So that means that if we started with even number of striped balls, then we will either reduce two or we will reduce zero balls. So every time we reduce two balls, we will, if at the end we have two striped balls, we will take these two and we'll put one solid ball back so the number of the striped ball will be zero in that case and the number of solid balls is one and that will be the outcome of the last ball.

In the second case when we started with an odd number of striped balls, so we reduced two striped balls in some steps and at the end we are left with one striped ball and some other solid balls. Now as we can see from our model, that we cannot remove one striped ball. We have to remove a pair of striped balls, two at a time. So that means if we are left with one striped ball at any point in time, that ball cannot be removed from the basket. And ultimately we'll be left with that one particular ball. And hence if we started with odd number of striped balls, we'll end up with a striped ball and if we started with even number of striped balls, then we end up with a solid ball in the basket. And hence the observation that we made earlier with the help of this mathematical model, we proved that that observation will hold whatever the number of balls that we play this game with. It does not matter. And the only thing that determines the outcome of the game is the number of striped balls and the number of solid balls they are immaterial, they are irrelevant as far as the outcome of the game is concerned.

This technique that we used here in this particular case, this technique is known as induction. What we do is when we are given a problem we start by applying the rules or

whatever that we have on small examples. So we started games of one ball, then we played the game with two balls. Then we played the games with three balls and we started to make some observations. And we are not really sure about how true those observations are, but at least we are getting some idea and that is what this induction is all about. And we can then prove things with that. And then in order to be sure about our observation, we tried to develop a mathematical model of the phenomena that we are trying to observe. And with the help of some kind of mathematical model we can hopefully prove the truthfulness of an observation. And that is what this thing is all about. I hope you enjoyed this game. I hope you learned something from this game. And I hope you will be able to apply this technique to solve some unknown problems that are given to you. This is a very useful technique. The basic principle here is when you are given an unknown problem and you are lost, start working with small examples. So start with the first step. So in this particular case the first step was we played this game with one ball. Then we played this game with two balls. Then we played this game with three balls. So when you do things like that, hopefully you will be able to make certain observations. And then once you have some observations you can draw some conclusions. We cannot say anything about those conclusions at that point but we have some idea. And then we try to come up with mathematical models that will help us in proving or disproving that conclusion that we have drawn and hopefully that will lead us to something that we can use later. So once again I thank you for having played this game with me. And I hope you enjoyed it. Goodbye and thank you once again.

Hi! So the first thing that you need to know about this game is that you can play this game with any types of objects. All you need is two different types of objects. You can have oranges and apples or anything that you can make a distinction between. So you don't necessarily have to have balls here. You could have anything. Now after the first break that we have, you should let the students play these games and the basic _____ here is let the students have this feeling of randomness. So if you play this game with 4,5,6 different balls, students will come up with different answers and then they will have this feeling that the answer is not deterministic here. And that is the first thing that I think will help them understand the process and understand this technique of induction, doing it systematically. So you play this game with the students and ask the questions and in most cases the students tell you that it is a random thing, it's not really anything that you can determine. It all depends upon the number of balls or the sequence of the balls or the sequence of the objects that you pull from the basket or bucket that you have. So you will get these kind of answers here.

And then we go to the second part, that is the second module and then there we make certain observations and we start looking at the problem systematically. Once again after we finish that, that is we finish playing a three ball game, some of the students might be able to come up with the right answer. But in my experience most students still need some more observations. So you play a four ball game with them. Remember that in a four ball game there are going to be five different possibilities. You can have all four solid balls, or you can have three solid balls and one striped ball. And then the third possibility is two of each type. Then one solid and three striped. And finally you can have four striped balls. So there are five different possibilities and you should tell the students, "OK you take this possibility and this is the

outcome." And so on and so forth. And hopefully after playing this four ball game the student will be able to make the observation. It is important that all these games are kind of note… you make notes of these games and you put them on the blackboard or something like that. That is, "OK we started with this thing. This is what happened. This is what happened and this is what the final result is." So when all the observations they have been listed down and enumerated. And then you put them in front of the students. That is the time when students really see the patterns there. If you just play a game and don't show these games all at the same time, the results of these games all at the same time, the students probably won't be able to make that conclusion or come to that conclusion there. So that is the second thing.

And the idea really here is that students that start with this kind of randomness, that think that this whole thing is random and then they start observing the pattern. So they realize that it is not really a random thing. So then we encourage the students to, "OK now you see there is a pattern. Can we actually convert that into a mathematical model?" Although you could have come up with the mathematical model without actually going through all this exercise in playing this game. But the idea really is once you start observing a pattern, that tells the student, OK there is some mathematics behind this thing. There is something that you can actually formulate and then use later. And that is probably the time when you introduce the students to come up with the mathematical models of such things.

So after we have finished this thing, you can encourage the students to play different types of games. There are a number of games that you can come up with which seem random at first but when you pay attention to these games, when you make observations, you can come up with the right answer and you can come up with the mathematical model.

So here is another game that you can play with the students and this game is typically played with a number of sticks but I'm going to play this game here in front of you with some balls. So the rule of this game is very simple. The rule is that I play this game with two players. So there are two players who are involved and playing this game. And we have a number of sticks, a number of objects. We know how many objects are there to start with. And these two players take turns in picking one, two or three objects at a time. So here I have let's say these 15 balls so I can pick one ball or I can pick two balls or I can pick three balls at a time. Players take turns in playing this game just like that. And the objective of this game is that the player who picks the last set of balls or objects that we have wins the game. So the question really is, given a number of objects can we tell who will win the game and what is the winning strategy? And if you let the students play this game and make certain observations and then come up with the mathematical formula and then they will realize that this game we can have a strategy based upon the observations the formula that we have and that strategy helps us winning this game no matter what the opponent does.

So let me show you how to play this game. I have these 15 balls here. So I am the first player so I pick 3 balls. The other player picks 1 ball. So when I pick 3 balls I'm left with 12 balls here. So the other player picks one ball so now I have 11 balls here. So I pick 3 balls once again. I can put these balls in the basket. So I am left with… the other player is left with 8 balls. This time he picks two balls. Now I'm left with 6 balls so I pick two balls again. Now the other player picks 1,2 or 3 balls, doesn't matter. Let's say he picks one ball. Now I can pick 1,2 or 3 balls so I pick the last set of balls and I win. Now is there a strategy to it or it was just a random thing and I was fortunate enough to win this game?

So you let the students play this game and you will see that once again if you start playing this game systematically that you play this game with one object. Then they play this

game with two objects. Then they play the game with three objects. Play this game with four objects. Play this game with five objects, six objects, seven objects, eight objects, nine objects and so on and so forth. A pattern will emerge. And pattern will be that if you started with a number of balls which are not divisible by four, and if you play the game correctly you will always win. And what is the mathematical formula behind this thing? How many balls should you pick to make sure that you win? The model as you will see, as you will observe, the model will come out to be very simple model. If N is the number of balls in the pile that I have, or number of sticks or number of objects that I have, I take _____ of this, that is I divide the number of balls, objects by 4 and I look at the remainder. And that is the number of objects that I need to pick in order to win the game. So I started for example when I started with let's say 7 balls. So these 7 balls, 7 divided by 4, the remainder is 3. So that means I need to pick 3 balls. So I will leave my opponent in a situation where the number of balls that are left is always divisible by 4. So whatever he picks, let's say he picks 2, I can always pick the other two. So I can always make it, convert it into a situation that the other player when his turn comes he is left with a number of balls which are divisible by four. So that means I will always end up winning the game.

So once again if you make your students, let your students play this game and make certain observations, hopefully they will be able to come up with this observation and they will be able to come up with this winning strategy.

You can change the rule of the game. Once they have figured this thing out, you can ask them, "OK now let us change the rule of the game and let us invert it." The person who picks the last object loses. So now what is the winning strategy? I want to leave one ball at the end so that the other person picks that ball and I win. So once again start playing this game, ask the students to play this game with one ball, two balls, three balls, four balls, five balls, six balls, and so on and so forth. And once again let them make certain observations and let them come up with a mathematical formula. And once again you will see in this particular case the formula is not going to be very different from what we have seen earlier. And it is going to be N + 1 divided by 4. And you look at the remainder and that is the number of objects that you need to pick in order to win the game. But once again you can let the students come up with the formula and help the student, guide the students towards that end. As I said, you can play a number of games. You can pick games from a different domain but continuing with this particular set of games, that is picking the objects and two players playing this game, you can change the rules once again. Another kind of rule could be that you ask the students, now the rule is that you can pick 1,2 or 4 objects, not 1,2 or 3, 1,2, or 4. So you cannot pick 3 objects. This is a challenging problem. And you can ask the students to take this home and play this game and come up with this model. And in this particular case you will see that there is a cyclic pattern. So the pattern is not very simple that you take this number of divide it by a certain number and that is the kind of remainder and the number of objects that you need to pick. It will be slightly more complex and slightly more difficult but you know if the students can actually discover that, that will be really great. So once again the idea really is let the students make certain observations and then based on certain observations let them come up with a mathematical formula. And so hopefully by doing this kind of exercise they will be able to apply these kind of principles on unknown problems that they come across and they will be able to figure things out. And that hopefully will open their minds to these new kind of problems that they see.

So once again I thank you very much for participating in the Blossoms project. I hope this exercise was useful and I hope your students will learn something on this exercise. Thank you very much.