Blossom Module-Probability Theory

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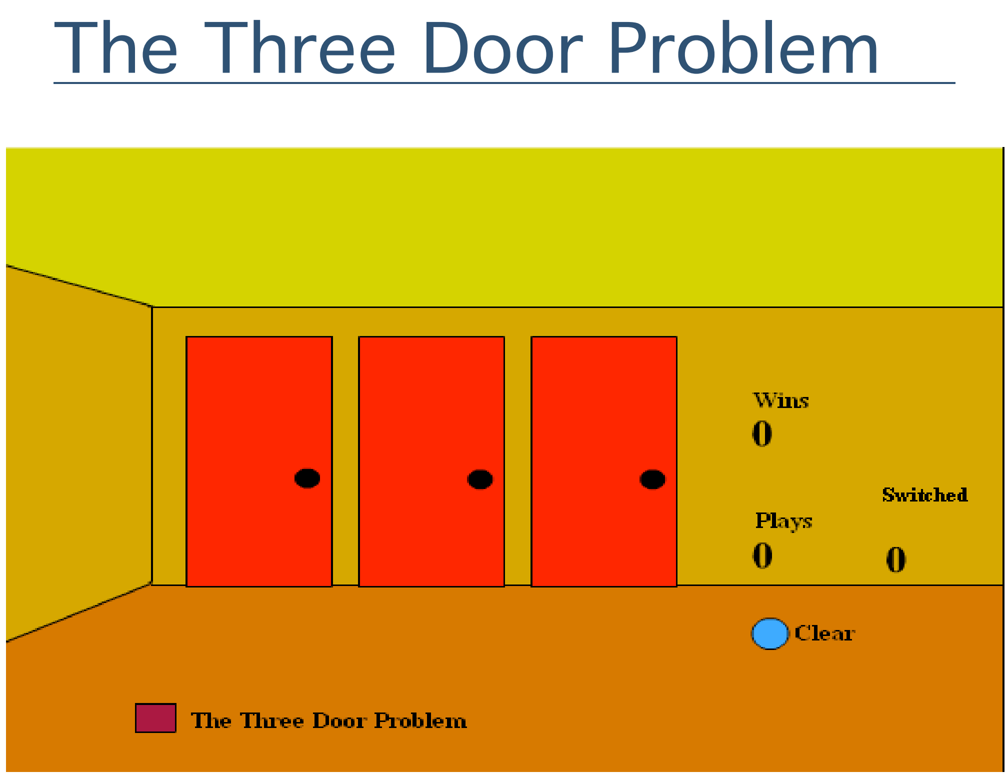
Section 1 (Introducing Case-Problem, 2:50 minutes):

Hello and welcome everyone! My name is Mohammad Al-Raqab, and I am a professor of mathematics at the University of Jordan, Amman, Jordan. This is my teaching assistant Mr. Alaa. He is going to help me in carrying some experiments in this class. I am keen to bring your attention today to the probability theory. My aim is to present the probability theory in a simple manner as possible. You probably know that the probability is a core of mathematical displines. You may know that the probability theory is a basic tool for handling an uncertain future and making a decision. The natural questions may arise here are as follows:
Can we use the probability theory to take good decisions and avoid wrong decisions?
Can one use the probability to choose the appropriate alternatives?
To see that, we need a "decision problem". So let's take a puzzling problem called 3-door problem. Imagine that there are three closed doors.



Behind one of the doors is a car, nothing behind the other two doors. Suppose the contestant chooses a door and the host who knows where is the car, opens one of the empty doors. The contestant is always given the option to switch his selection. That is; the contestant has two strategies: Strategy 1: Always stick with his first choice; Strategy 2: Always switch doors.

This game is also named “Monty Hall Problem”. In fact this game comes from the popular american game show “Let Make A Deal” in the 1960’s. After capturing public attention, it has reappeared in both academic work and the popular press.
Now, what do you think? Which strategy the contestant should follow? This will keep you busy next five minutes. Discuss the three-door problem with your classmates. See you soon.

Section 2 (Presentation of the Case-Problem, 3:09 minutes):

Welcome back everyone! Which is the good decision? Do you think it is better for the contestant to stay with his first choice or switch to another door? Let’s make the three-door problem more realistic?

Now we have three marked envelopes, labeled A, B and C. Inside one of these envelopes, 20 JD. I invite my teaching assistant Alaa to place the three envelopes on the table. Now I am asking Miss Sara, one of mathematics students to come over here and pick one envelope. Now Sara has chosen Envelope A. I am asking my teaching assistant Alaa to choose an envelope. Let us ask Sara whether she is going to change her mind. It looks that Sara up to this moment she sticks with her choice. Suppose I am give you further information by inviting all of you to partial videotape...

So Alaa always opens an empty envelope; he never opens Envelope Miss Sara initially chose. Mr. Alaa chooses randomly from among the other two envelopes. Now we are given your classmate Sara an option either she can stay with her choice or switch to another envelope.... Now Sara informed us she is going to change her mind and change the envelope to Envelope B where 20 JD is placed there. Therefore the best strategy for Sara is to change her choice and choose Envelope B.

To get an idea about the probability value p, the class would be divided into 3-4 groups and each group would run the three-envelope game 12 times either by staying with the first choice or switching into another choice. I am sure you will be able to figure out which probability has a high value? Probability of winning if you switch the first choice or the one if you stay with your first chice!

I am looking forward to hearing your response after a five-minute break. I will get back to you soon.

Section 3 (Determination the Probability Value, 6:34 minutes):

Welcome again! Were you able to come up with the probability value p? Well, can you identify all the different possibilities that could occur? You may observe that the value of p depends on your option whether you are staying with your parent choice or switching to another envelope. Consider that the Sara selects Envelope A and my teaching assistant Alaa opened Envelpope C.

Does it matter if you change your mind?
What is the value of p if you stick with your first choice (Envelope A)?
What is the value of p if you decide to switch to envelope B?

Surprisengly, the answer to quation # 1 is yes, it does matter. To see this, let us enumerate all possibilities. For the first three games; you choose Envelope A and "stay" each time ؛Here are the results.

	Envelope			
Game	A	B	C	Result
1	20 JD*	---	---	stay and you win
2	---	20 JD	---	stay and you lose
3	---	---	20 JD	stay and you lose

In Case 1, my assistant Alaa opens either Envelope B or C, if Miss Sara sticks with her choice then she will get the money. If Sara switches to another envelope, she will get empty envelpe. In Cases 2 and 3, Sara will loose the money if she changed her parent decision. If Sara sticks with her first choice, she only wins when her initial guess was correct. For the second three games; you choose Envelope A and "switch" each time. We also can enumerate all cases. Here are the results.

	Envelope			
Game	A	B	C	Result
1	20 JD*	---	---	switch and you lose
2	---	20 JD	---	switch and you win
3	---	---	20 JD	switch and you win

In Case 1, the money may be in Envelope A and nothing in B or C. This means that in this case it is better to stay with your first choice. In Case 2, the amount is placed in Envelope B and nothing in A or C and changing the choice implies winning the money. In the last case, the money may be placed in Envelope C and in this case it is better to change the choice and get the money. In otherwords,

$$P(\text{winning 20 JD if Sara sticks with her choice}) = 1/3$$
$$P(\text{winning 20 JD if Sara switches her choice}) = 2/3$$

Let us repeat the problem but with 5 envelpes where 20 JD is placed in one of these envelopes. I am asking Mr. Alaa to place the 5 envelopes on the table. Could Miss Sara come over here and select one of the envelopes. Now, I will ask my assistant Alaa to dramatically open three empty envelopes, leaving Sara with her first choice and the remaining un opened envelopes. Similarly, Miss Sara has two options, either she can stay with her first choice or switch to another envelope.

In this case, we will have 5 choices and there is one choice of being the money inside an envelope. As a result, we have

$$P(\text{winning 20 JD if Sara sticks with her choice}) = \frac{1}{5} = 20\%$$

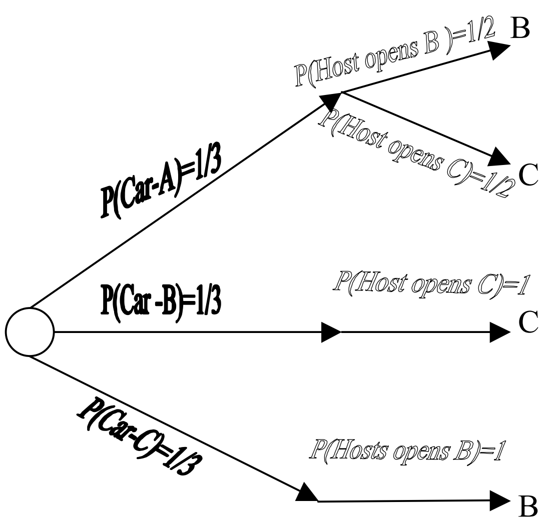
$$P(\text{winning 20 JD if Sara switches her choice}) = \frac{4}{5} = 80\%$$

The main thing that you can do now is to be able to write down all possible outcomes of the envelopes. Now, think about that with your teacher. Do these probability values make sense? I do hope that you will figure out these numbers. I will get back to you soon!

Section 4 (Simple Representation-Probability Tree, 4:43 minutes):

Welcome back! The major task here is whether we can have a simple representation for computing the value of p even for other similar problems. Actually, there is a simple technique called Probability Tree. Our main problem and its respective probabilities can be summarized and prsented using the following table.

Marginal Prob. **Condition Prob.**



Joint Events	**Joint Probability**
Car-A and open B	1/3 x ½ =1/6
Car-A and open C	1/3 x ½ =1/6
Car-B and open C	1/3 x 1 =1/3
Car-C and open B	1/3 x 1 =1/3

Clearly, the probabilities on the stems are marginal probabilities. In otherwords, P(Car is behind Door A)= P(Car is behind Door B)= P(Car is behind Door C)=1/3. The probabilities on the leaves (second branches) are called conditional probabilities since they are conditioned on the first branches! Do these make sense? Now, multiplying the probabilities on the stems by the probabilites on the leaves, we obtain joint probabilities. For example, multiplying the probability on the stem A which is 1/3 by the conditional probability on branch B which is ½ , we get (1/3)(1/2)=1/6. The samething for Stem A and Branch C we get (1/3)(1/2)=1/6. When multiplying the probability on Stem C, 1/3 by the probability on Branch B, 1 we obtain (1/3)(1)=1/3. Finally, multiplying the probability on Stem B, 1/3 by the probability on Branch C, 1 we obtain (1/3)(1)=1/3. These resulting probabilities are joint probabilities. So we have three types of probabilites: marginal, conditional, and joint probabilities.

I hope that the teacher and students can discuss the relationship between the marginal, conditional and joint probilities during the break and then I can see you again!

Section 5 (Computations-Probability Tree, 9:12 minutes):

Hello again! Now from the Probability Tree, we can easily determine the following probabity values: P(winning if the contestant sticks with his parent choice) and P(winning if the contestant switches to another door). Now, Let us assume that the contestant has chosen Door A. To address the problem logically, we first draw up a 2x3 probability matrix showing the host's response to the contestant's choice. Could you please help in filling out the cells of the matrix based on the Probability Tree.

	Car at A*	Car at B	Car at C	Marginal Probilities
Host opens B	1/6	----	1/3	1/2
Host opens C	1/6	1/3	----	1/2
Marginal Probilities	1/3	1/3	1/3	----

There are 3 cases. The car can be either behind Door A or B or C. The host opens an empty door. The probability that the contestant has chosen Door A and the host opens Door B is 1/6 (joint probability). Same probability will concluded if the contestant opens Door A and the host opens Door C. If the car is behind Door B then the host will not open Door B and the probability the host will open Door C is 1/3. If the car is behind Door C then the host will not open Door C, he is going to open Door B with joint proability 1/3. By summing the probabilities on the first column we get the marginal probability that the car is behind Door A which is 1/3. The sum of probabilites on the second column is 1/3 representing the probability that the car is behind B and the sum of probabilities on the third column is 1/3 representing the probability that the car is behind Door C. Also we have other marginal probabilities. The probability that the hos opens Door B is 1/6 + 1/3= ½ and the probability that the host opens Door C is 1/6 + 1/3= ½.

Based on the joint and marginal probabilities, we can compute the following conditional probabilities:

$$\text{P(Car-A | Host opens B)} = \frac{P(Car - A \text{ and } Host\, opens\, B)}{P(Host\, opens\, B)} = \frac{1/6}{1/6 + 1/3} = \frac{1}{3}$$

P(Car-A | Host opens C) = $\dfrac{P(Car-A\ \ and\ \ Host\ opens\ C)}{P(Host\ opens\ C)} = \dfrac{1/6}{1/6+1/3} = \dfrac{1}{3}$

P(Car-A| Host opens any door) = $\dfrac{1}{3}$

P(Car-B | Host opens C) = $\dfrac{P(Car-B\ \ and\ \ Host\ opens\ C)}{P(Host\ opens\ C)} = \dfrac{1/3}{1/6+1/3} = \dfrac{2}{3}$

P(Car-C | Host opens B) = $\dfrac{P(Car-C\ \ and\ \ Host\ opens\ B)}{P(Host\ opens\ B)} = \dfrac{1/3}{1/6+1/3} = \dfrac{2}{3}$.

Consequently,

P(Winning the car if Contestant sticks with his choice) = 1/3

P(Winning the car if Contestant switches his choice) = 2/3

I hope you enjoyed learning the probability theory and you can figure out the relationships between the marginal, conditional and joint probabilities. I wish you best and good luck in your studies. Bye!

Teacher Guide Section (7:29 minutes):

Hello. This is Mohammad Al-Raqab again and I am talking to you as the high school teachers. I am now willing to explain some ideas and thoughts to be considered throughout the four breaks. The main objective of this lesson is to motivate students thoughts and get them excited with some probability concepts.

In the first break, it is extremely important for the student to figure out the connection between this case problem (three-door problem) and the probability theory. Within this break, after I introduce the case problem, which is the three-door problem, the teacher may lead the class by focusing on two issues. The first issue is to convince the students that the probability helps them figure out the likelihood of something happening. The second issue is to ask students to present other similar problems. Acutally, the three-door problem was a very popular show and surely there are other related problems in which the probability theory play an important role in getting a good decision.

Break # 2 allows the teacher and students to interact more effectively and touching the best choice. The high school teacher would ask the students to run the three-envelope experiment multiple times switching and sometimes staying, and see what happens. Precisely, the students can play 12 times using Strategy 1 (sticking with the first choice) then 12 times using Strategy 2 (switching doors) and then compare the results which would be presented in the table given below. Check whether either strategy is better.

Case	Classmate Selection	Assistant or Teacher Selection	Decision of the Classmate	Win/not Win
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				

Based on a real experiment, it is easily observed that it is preferable to switch when given the chance

With regard to Break #3 after presenting the problem in a realistic situation as well as having observed the possibilities of the outcomes, students are expected to be engaged more in getting an exact solution for the probability values. One way to draw the student's attention to the correct answer is asking them to enumerate all the possibilities taking into account that there are two stages. It is extremely important if you as a teacher can ask the students to enumerate the possibilities of the 5-envelope experiment and figure out the reasoning of the answers given in the class.

			Envelope			
Case	A	B	C	D	E	Result
1	20 JD	---	---	---	---	Stick and Win
2	---	20 JD	---	---	---	Stick and Loss
3	---	---	20 JD	---	---	Stick and Loss
4	---	---	---	20 JD	---	Stick and Loss
5	---	---	---	---	20 JD	Stick and Loss
1	20 JD	---	---	---	---	Choose and Loss
2	---	20 JD	---	---	---	Stick and Win
3	---	---	20 JD	---	---	Stick and Win
4	---	---	---	20 JD	---	Stick and Win
5	---	---	---	---	20 JD	Stick and Win

This will guide the students to the right methodology in getting the probability numbers and making the correct decision. Therefore

$$\text{P(Wining in case of stick with the choice)} = \frac{1}{5} = 20\%$$

$$P(\text{Wining in case of changing the choice}) = \frac{4}{5} = 80\%$$

Break # 4 should mainly focus on the likelihood of an event or the probability value of an event. Let me suggest a simple example here the teacher can discuss with his students. Consider a jar with three colored balls (blue, yelow, green) such that the jar contains 5 blue, 3 yelow, 2 green balls. The marginal probabilities can be observed by obtaining the chance of getting a blue ball, or yelow ball or green ball. Specifically,

$$P(getting\ a\ blue\ ball) = \frac{5}{10} = \frac{1}{2}$$

$$P(getting\ a\ yelow\, ball) = \frac{3}{10}$$

and

$$P(getting\ a\ greenball) = \frac{2}{10}.$$

If the first drawn ball is blue, what is the chance of geeting a blue ball in the second trial (you draw without replacement). This means that P(2nd draw is blue| the first draw is blue)= 4/9. This is a conditional probility. Now joint probability of getting blue ball in the first draw and blue ball in the second draw)=(5/10)x(4/9)=2/9. The first probability (5/10) is a marginal probability and the second one (4/9) is conditional probability and its value is based on the first drawing. The joint probability should be changed if the drawing is with replacement. That is, once you draw the first ball you place it in the jar before drawing the second ball. In consuence, the probability of getting blue balls in the first and second draws = (5/10)(5/10)=(1/4). These facts will help the students to get a definite number for the probabilities of the events raised in this lesson as they computed in the following section.

I wish you best and hope that I can meet you in another meeting.