BLOSSOMS FABULOUS FRACTALS
By Laura Zager

Hi. My name is Laura Zager and today we're going to be exploring an incredibly beautiful and fascinating area of mathematics using algebra and complex numbers.

Let's get started with an experiment called "the chaos game." On the board behind me, I have an equilateral triangle with the three vertices labeled "red" and "blue" and "green." And here on the table I have a six-sided die with two faces colored red, two faces colored green and two faces colored blue.

Here's how you play the game. Start by picking one of the three vertices. I'm going to pick the blue one, but you could pick any one you'd like. Draw a dot at the vertex that you've chosen. This is going to be your starting point. Next, roll your die and observe which color comes up. So I've just rolled a green. So what I'll do is I'll start with my blue point that I've just chosen. I'll look at the green corner and I'll draw a new dot on the triangle halfway between my current point and the green corner. So if I measure roughly halfway, maybe I'll be here, and I draw a new point on my triangle. Now this is my new current point and I'm going to repeat this procedure again. I'm going to roll the die and I've gotten a red face. So I'll draw a new point on my triangle halfway between my current point and the red corner. About here. I keep repeating this procedure, drawing lots and lots of dots. Rolling the die, drawing a new dot halfway between my current point and the vertex corresponding to the color of the die.

Now here's a question for you. If I keep doing this over and over and over again, drawing more and more dots, what do you think the pattern of dots in the triangle is going to look like? Or will there be no pattern at all? Do some brainstorming. Collect your guesses and we'll come back to this at the end of the lesson.

Now let's change direction a little bit. Have you ever heard of the Fibonacci numbers? The Fibonacci numbers are a sequence of numbers defined by a simple rule. The first Fibonacci number x0 is just defined to be zero. The second Fibonacci number x1 is defined to be one. The rest of the Fibonacci numbers are defined to be the sum of the previous two Fibonacci numbers. So x2 will be the sum of x1 plus x0 which is one plus zero, just one. x3 will be the sum of x2 plus x1, x2 plus x1 gives us two and so on. And we can compute any number, x3, x4, x5. The Fibonacci numbers have interested mathematicians for over a thousand years, both because of their interesting mathematical properties and also because their tendency to arise in all sorts of natural patterns, like the golden spiral found in nautilus shells and in the shape of galaxies.

Let's come up with a more mathematical way of writing down the Fibonacci number rule. If we want to find the nth Fibonacci number xn, first we have to take into consideration our two special cases. When n=0, we know that the x0 Fibonacci number is just zero. When n=1, we know that x1 is equal to one. But for every other value of n, n>1, the nth Fibonacci number is just the sum of the previous two. The nth term will be the sum of n-1th term and the n-2th term. We have special names for the relationships that are in this equation here. This relationship is called a difference equation. A difference equation is simply a rule that tells us how to get future items in the sequence from the items that already computed. These two relationships in our definitions have a special name as well and they're called initial conditions. Initial conditions tell us how to get started applying our difference equation. For the rest of this lesson today, we're going to be looking at difference equations that only require one initial condition. That initial condition we'll always call x0. If we also have a difference equation, then we can use the difference equation, apply it to x0 to compute x1. Apply the difference equation to x1 to compute x2 and so on, computing as many terms in the sequence as we'd like. If we have a set of terms that satisfy a difference equation and an initial condition, we call that sequence of terms a trajectory. So just like following the path of a ball across the sky, that's following its trajectory, looking at the sequence of values that a difference equation takes is also called a trajectory.

Let's look at an example together. Consider this difference equation. $X_n=2x_{n-1}+1$. And what we'll do is come up with a few different initial conditions and see what different kinds of trajectories result. Let's try for our first initial condition $x_0=1$. Well, if we use $x_0=1$ as our initial condition, 2 times 1 plus 1 gives us the next term in the sequence so it would be 3. 2 times 3plus1 gives us 7. Two times 7 plus 1 gives us 15 and so on. We could keep computing terms in this trajectory if we'd like.

Let's try another initial condition. Let's try $x_0=0$. Well, 2 times 0 plus 1 gives us 1. 2 times 1 plus 1 gives us 3. 2 times 3 plus 1 gives us 7 and so on.

What about another initial condition? Let's try –1. 2 times -1 equals -2. We add 1 to that, we get –1 back out again. Well, if we apply the difference equation again, 2 times -1 plus 1 also gives us –1 and so on. So this trajectory will always stay the same. If we start with –1 we'll always get –1 out as a result. Let's try –2. 2 times -2 is –4, plus 1 equals -3. 2 times -3 equals-6, plus 1 equals -5. And 2 times -5 plus 1 gives us –9 and so on.

So are there any patterns in these trajectories at all? Well, just from these experiments, it looks like when our initial condition is greater than minus one, then the trajectories tend to get more and more positive. So if we continue computing the elements in this trajectory they would just keep getting bigger and bigger. Similarly, when our initial condition was less then –1, it looks like the trajectory became more and more negative the further we went. But when our initial condition was exactly –1, the trajectory always stayed the same; it always stayed at –1.

Here are a few more difference equations for you to try with your class. Choose some initial conditions, compute a few trajectories and see what kinds of interesting patterns you can discover. Collect your answers and we'll come back together.

In the examples that you just explored, you saw that some trajectories tended to head off to infinity, either positive or negative. And some trajectories tended to stay finite. What do we mean when we say that a trajectory heads toward infinity? What we mean is that its absolute value tends to grow without limits or bounds. Let's try to come up with a mathematical way of saying this. When do we call a trajectory "bounded"? If we can find a single number b, such that the absolute value of every point in that trajectory is always less than b. Even if the b that we find is a really big number, like 10 million, if every point in our trajectory is always less than 10 million, we call that trajectory a "bounded trajectory." If a trajectory isn't bounded, then it's unbounded. What does that mean? Well, it means that we can't find any such single number b such that every point in the trajectory has absolute value always less than b. Another way of saying this is that for any value b we choose, you can always find some point in the trajectory whose absolute value is greater than that number.

Let's look at the example that we explored earlier. The difference equation $X_n=2x_{n-1}+1$ Which of the trajectories that we computed earlier is bounded and which is unbounded? Well this first trajectory seemed to be continuing to grow. And in fact. every time we find a new term it's going to be a little more than twice as big as the term before. That means this trajectory heads toward positive infinity and this is in fact unbounded. The same is true for the second trajectory. These terms will continue to grow towards positive infinity. What about this last trajectory? These terms are getting more and more negative but their absolute values are getting bigger and bigger. And they'll continue to get bigger and bigger. This is also an unbounded trajectory. In fact the only bounded trajectory of this difference equation is this one:

when we start with the initial condition -1 and get –1 for the entire trajectory. Practice with the definitions of bounded and unbounded a little bit more. Go back to the examples that you just worked in your class. And decide which of the trajectories that you computed are bounded and which are unbounded.

Now that you're all trajectory experts, let's consider a new difference equation. Xn=(xn-l)^2 +c where c is a constant that we haven't specified yet. This time though we're going to add a twist. We're going to let c be a complex number, not just the real numbers that we've been considering so far. Remember that a complex number has both a real part and an imaginary part. So in order to work with difference equations that are complex, we're going to need to know how to multiply and add complex numbers. When looking at this difference equation, let's assume that our initial condition is always x0=0. We'll be interested in the different trajectories that result for different choices of complex c. Let's consider one example together. If we choose c=0, if x0=0, our difference equation is simply xn=(xn-1)^2 and the result there is 0 and 0 squared is still 0. And will continue to be 0. So in fact this entire trajectory is all 0's and this is indeed a bounded trajectory.

Now as a class consider these two possible trajectories: the choice when c=1 and when c=i, where i is the imaginary number. Now, in order to decide whether these trajectories will be bounded you're going to need to be able to take the absolute value of a complex number. But that you already know how to do. So with your class, decide if a few trajectories are bounded or unbounded and we'll come back together and move on to something exciting.

I hope you found that when c=1 the trajectory was unbounded and when c=i the trajectory was bounded. Now here's an idea. What if we used the boundedness or unboundedness of these trajectories as a way of coloring in the entire complex plane? Here's what I mean. Here I've drawn the complex plane: the real axis and the imaginary axis. Here's our coloring scheme. For any point c in the complex plane we'll color that point c red if the difference equation xn=(xn-1)^2 + c is bounded when our initial condition is x0=0. And we'll color that point green if xn=(xn-1)^2 + c is unbounded with the initial condition x0=0. So, for example, we've already computed a few of these. Let's fill them in. We found that when c=0 and c=i, these trajectories were bounded. That means we'll color these points in red. Here is c=0 and here is c=i. c=1 corresponded to an unbounded trajectory so we'll have to color in c=1 green.

What we'll do next is take all of the red points and compile them into a set we'll call M which stands for the Mandelbrot set. The Mandelbrot set was invented by Benoit Mandelbrot, one of the first mathematicians to be interested in this set. Now here's my question for you. If we were to consider every single point in the complex plane and color it either red or green according to whether or not its resulting trajectory was bounded or unbounded, what do you think the pattern of points would look like? Or would there be any pattern at all? Take some time to brainstorm about this with your class and we'll come back together with a really surprising answer.

In fact, the Mandelbrot set looks like this. What a surprising and complicated shape! Let's zoom in on the boundary of this shape just to see what happens. You'll notice that as we zoom the piece that we see looks just as complicated as the original shape. Indeed we'll also see repetitions of the original shape, of the larger outline appearing the more that we zoom. This property is called "self-similarity." In fact, no matter how far we zoom, the boundary of the Mandelbrot set will always be just as complicated and will never simplify. Because of this we say that the boundary of the Mandelbrot set is a "fractal." The word "fractal" was coined by Benoit Mandelbrot in 1975 when he noticed his computer producing some very interesting images according to the difference equation that we just studied. One of the things that makes the study of fractals so fascinating is that they appear regularly in nature, like in this cauliflower and in this cactus. Artists also love to work with fractals, using their computers to create incredible images based on difference equations.

One of the most famous fractals is also the simplest. It's called the Sierpinski triangle. Here's how you build a Sierpinski triangle. First, start with a black equilateral triangle and remove an upside down triangle from the middle, leaving three black triangles. Now remove an upside down triangle from the three remaining black triangles and continue doing this for every remaining black triangle. And continue this whole process an infinite number of times. This set is a fractal because of its self similarity. No matter how we zoom in on it, we'll always see the same thing. Removing smaller triangles from larger ones isn't the only way to generate the Sierpinski triangle. You remember the chaos game that we played at the beginning of this lesson? Well I've written a little computer program to play it for us fast. Let's see what happens. Yes, it's the Sierpinski triangle! The relationship between the Sierpinski triangle and the chaos game is an incredibly deep and interesting one, one that I hope you're interested in pursuing after this lesson is over. I hope you've enjoyed learning a little bit about fractals today. Thank you so much for your time and for your energy.

(Teacher Guide Segment)

Hi there. I hope you're looking forward to this lesson. It's really been a joy to teach in my classroom, and I hope it's a joy to teach in yours as well.

I have lots of things that I want to tell you, so please pardon my notes as we go along.

The lesson on fractals was designed for students who have a mastery of basic algebra skills. So, in the US that often means maybe a 10th grade or 11th grade level. That's the level that I've taught it at in the past. It also requires a knowledge of complex arithmetic, which is not a topic that gets taught in all algebra classes.

So to achieve that topic and prepare the students for the lesson, we've actually written a tutorial on complex arithmetic. This tutorial can be used in a number of ways if your curriculum doesn't cover complex numbers. One way to do it is to assign it to your students as homework the night before you are to do the lesson. It's designed to start from the basics, for a student who knows algebra but knows nothing about complex numbers and bring them up to the level where they can add, subtract, multiply, and take the absolute value of complex numbers. So it walks them through these steps and it gives them some practice with it.

Alternately, you could decide to actually work through the tutorial in your classroom. So take the day, or a few days before you decide to do the fractals lesson and give your students time in a group, and with you, to get comfortable with the mechanics of complex arithmetic before you launch into this material.

We've also included in the written material that goes along with this lecture a very quick quiz on complex arithmetic. Depending on your class, maybe you'd want to use it the morning that you do this lecture, or assign it in advance, just to check the comprehension of the students before you launch into the more complicated material.

So that being said, the actual mechanics of the math that's in this lesson are not too hard. They're addition, subtraction, and multiplication of numbers. What's difficult about the lesson is that it introduces some really sophisticated new concepts for students that require a lot of mathematical maturity: difference equations, initial conditions, trajectories, boundedness. These are all really quite sophisticated ideas. So you're going to play a really important role in helping your students understand and feel comfortable and develop intuition about these new topics. I hope you are up for the challenge.

OK, let's get started.

Part I of the lecture is an experiment called the "Chaos Game." In the video, I present it and suggest to the students that they pursue it as a thought experiment. So the way I've done it in my classroom in the past is broken the students up into pairs and had them work together and brainstorm what they think the final pattern of dots is going to look like. Often, it helps to hand each pair of students a piece of paper with an equilateral triangle already drawn on it, and that way they can plot out some points and look at it and do some discussing.

Alternately, you might want to try actually running this experiment in your classroom. Robert Duvaney is an educator who suggested one way to do this is to hand small groups of students, or individual students, an overhead transparency sheet – so, a clear sheet of paper. You also hand them a permanent market, and some kind of die or way for them to run the experiment. So some way of selecting red/green/blue, or three equally likely things. So if you had a die, you could use a die. You could make your own, like the one I used in the video. If you wanted to, you could draw three different colored marbles out of a bag or hat. Any of these ways would work just fine. So if you hand each of your students this overhead transparency sheet with a triangle drawn on it, have them actually run this experiment and draw dots on the overhead sheet. Duvaney suggests that when you're finished with this process, you can take all the transparency sheets and stack them up and then project through them onto the wall. The idea is that this is a very quick way of generating a whole lot of points. You should be able to get a sense of what the final pattern looks like.

I think in order to make this successful you'd have to have some really, really precise students, because when I tried this with a group of teachers in the past, the result was a completely incoherent mess. We couldn't see any patterns at all. So I recommend doing some experimenting with this before you get started, before you bring it into your classroom.

An alternate way of doing it would be to actually write a computer program that would do it. At the end of this lesson, in the very final section, we present the results of a computer simulation that's playing this game. If you have students in your class who are taking a computer programming class, or who know something about computer science, this might be a fun project for them to do—have them write a little piece of software that would play this game for the whole class. It could be a really interesting extension.

OK. Speaking of extensions, you may get questions on extensions. For example, what happens if you don't start the game at one of the vertices, but you start at some point in the middle of the triangle. Are you still going to get the same kinds of patterns? These questions are what makes it a lot of fun to be a teacher. So the more thinking you can do about the kinds of questions your class is likely to ask, and the answers to them, or at least giving them pointers to where they can explore the answers to the questions themselves, could be really great.

The written material that accompanies this video lecture includes a lot of possible extensions. So I hope you will take a look through those and maybe get some of your more excited students to pursue some of those topics.

So that's Part I, playing the Chaos Game.

Part II is the introduction to difference equations. Right away, some of your students may have trouble with this because a difference equation is very different from the kinds of equations they are probably used to seeing. In a lot of ways, it's more like a rule than some kind of formal equation they're going to have to solve.

We start by trying to develop students' comfort with this idea of difference equations by talking about the Fibonacci Sequence. Often in my classroom, there have been one or two students who have seen it somewhere else in their previous background. But many students haven't. It's a really great topic, just the Fibonacci numbers themselves, for high school algebra and geometry classes, because there is a very strong geometric component to the Fibonacci numbers. They have a lot of interesting applications, and there are a lot of interesting and not-too-complicated mathematics that can really give high school students a glimpse into what mathematics at the college level might be like without bogging them down in too much material that they don't have the background for. So I highly recommend encouraging your students to pursue different areas of applications of the Fibonacci numbers. And there are some discussions of that material in the written material that accompanies this lecture.

This section introduces difference equations and initial conditions. Really, what's needed more than anything here is just time and practice. Students need to get familiar with looking at a difference equation, being comfortable computing a trajectory, and experimenting with what happens with different initial conditions.

A really useful format that you can apply to the problem-solving that happens throughout this entire lecture, is to break the students into small groups, present them with the examples that we present in the video, have them work in these small groups for a little while, and then bring the entire class back together and assemble all the results so everybody can see them. So if you can put them all up on the board, that's a really useful thing. And that's especially useful when you're talking about difference equations and calculating example trajectories, because then you can get a lot of different kinds of trajectories without having every student have to go through all the different possibilities.

In Part III, we introduce the idea of boundedness. I think this is one of the more sophisticated ideas that gets introduced in this lecture. I think you'll probably end up spending a lot of time helping students develop intuition on the idea of boundedness.

Often, I find it helpful to phrase boundedness in the following way: if a trajectory is unbounded, then it's not possible to find any single number such that the absolute value of the trajectory is always less than that number. So if you can find such a number—even if it's a really big number, even if it's 100 million—if you can find a number such that the absolute value of the trajectory is always less than that number, then it's bounded. You'll come up with all sorts of different ways of explaining what a bounded trajectory is versus an unbounded trajectory.

A useful tool to use in the classroom is to encourage your students to challenge each other. So present a difference equation, and then an initial condition. Look at the trajectory, calculate a few points of it, and then ask the class: "Is this bounded or unbounded?" Often they will have an intuitive answer: "Well, it looks like it's blowing up," or "It looks like it's not." See if you can get them to use the mathematical language. Ask them to prove it to you.

"If you argue that it's bounded, then find me a number that the absolute value is always smaller than. If you argue that it's unbounded, then tell me why."

Another issue that often confuses students is when a trajectory is bounded, a number that serves as a bound to that trajectory is not unique. So, for example, if the absolute value of a trajectory is always less than 2, then the absolute value is always less than 3 as well. Students often find it difficult to handle this non-uniqueness in terms of boundedness, so that's something you may have to discuss explicitly in your class.

In Part IV, we add a twist to the material we've done so far. We allow our difference equations to take on complex values. So maybe we give them complex initial conditions, or the difference equations themselves have a complex number in them. None of the mechanics are any different, but it requires them to use complex arithmetic. And depending on how comfortable your students are with complex arithmetic, this can be more or less difficult. This is another one of those topics that just requires time and practice. So having extra examples, and allowing enough time for students to work through those examples can be really important to helping them feel comfortable with the idea of complex trajectories.

In order to evaluate whether or not a complex trajectory is bounded, students need to be able to take the absolute value of a complex number. This is a topic that is addressed in the tutorial, but if you discuss complex arithmetic in your class outside of the tutorial, make sure the students know how to find the absolute value of a complex number in order for them to be able to work through this section.

Part V introduces the Mandelbrot Set. In some ways, this is conceptually the most lovely part of this entire lecture. At this point, students need to feel comfortable with deciding whether or not a trajectory, a complex one, is bounded or unbounded because that's the way you define what points are in the Mandelbrot set or not. But hopefully, by this point, all the hard work and heavy lifting has been done by your students in terms of the mechanics and intuition-building on trajectories. Now, they can just enjoy the beautiful results that come out.

So at the end of this section, we ask the students to guess what the Mandelbrot set is going to look like. This is one of those times where you don't need to spend too much time in class on having students brainstorm and then coming back together with their suggestions. Because the answer is so strange, and seemingly unpredictable, it's really more of a thought exercise at this point, and they are kind of waiting to see what the final answer is going to be.

The very last section is on fractals in general. It is really just a brief introduction to some beautiful images that will hopefully inspire your students to pursue some of the extensions that we were talking about earlier, and to do future work in this field.

At the very end, we also return to the chaos game and the Sierpinski triangle. As I mentioned earlier, we present the brief computer simulation, which shows you the very surprising and interesting pattern that the chaos game will result in.

So I hope by the end of this lesson, your class has learned an extraordinary amount of new material (most material that students in high school never get a chance to see) and that all of you have become interested in a brand new topic that could take you in all sorts of directions in algebra and geometry and beyond.

Thank you very much for your interest in this lesson. I hope you enjoy making this lesson on fractals exciting and interesting to your class, and I wish you the best of luck.