Interpretation of β in log-linear models

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1 Model

Our econometric specification for the relationship between x and y is

$$\log(y) = x\beta + \varepsilon$$

We are interested in the interpretation of β, specifically, when does β mean that a one unit change in x is associated with a $100 \cdot \beta\%$ change in y?

2 Approximate $\%\Delta y$

As $x_0 \to x_1$, what happens to y in percentage terms? In other words, what can we say about

$$\equiv \frac{y_1 - y_0}{y_0}$$

in relation to

$$\Delta x \equiv x_1 - x_0$$

(where conventionally we think about $\Delta x = 1$)

Well, let's start with what we know about $\log y$. We know that for a change Δx, the corresponding change in $\log y$ is

$$\begin{aligned}
\Delta \log y &= \Delta x\beta \\
&= \log y_1 - \log y_0 \\
&= \log\left(\tfrac{y_1}{y_0}\right) \\
&= \log\left(\frac{\Delta y + y_0}{y_0}\right) \\
&= \log\left(\frac{\Delta y}{y_0} + 1\right) \\
&\approx \frac{\Delta y}{y_0} \equiv \%\Delta y
\end{aligned}$$

Where the approximation (from a Taylor Series expansion around $z = 0$) that $\log(1 + z) \approx z$ for small z was used in the last step.

Thus, a change Δx is associated with approximately a $100 \cdot \Delta x\beta$ *percent* change in y.

3 Exact $\%\Delta y$

First, let's write down the exact quantity that we want to examine for a given change in x

$$
\begin{aligned}
\%\Delta y &\equiv \frac{y_1 - y_0}{y_0} \\
&= \frac{y_1}{y_0} - 1 \\
&= \exp(\log(y_1/y_0)) - 1 \\
&= \exp(x_1\beta + \varepsilon_1 - x_0\beta - \varepsilon_0) - 1 \\
&= \exp(\Delta x\beta + \Delta\varepsilon) - 1
\end{aligned}
$$

So, if we are interested in the percentage change in y for a Δx change (e.g. $\Delta x = 1$) in x (ceteris paribus, holding $\Delta\varepsilon = 0$), then the exact percentage change in y implied by our log-linear model is

$$\boxed{\%\Delta y = \exp(\Delta x\beta) - 1}$$

4 Comparison of log points and percentage points

The approximation in Section 2 used the fact that $\Delta y/y_0$ was small, which is likely to be the case for a small quantity $\Delta x\beta$. However, using the log point change in y implied by β as the approximation to the percentage point change in y always gives a biased downward estimate of the exact percentage change in y associated with Δx.

For example, if $\hat{\beta} = .3$, then, while the approximation is that a one-unit change in x is associated with a 30% increase in y, if we actually convert 30 log points to percentage points, the percent change in y

$$\%\Delta y = \exp(\hat{\beta}) - 1 = .35$$

So instead of a 30% increase as suggested by our approximation, the exact percentage increase implied by our estimate is 35%. The approximation is a lower bound.

If $\hat{\beta} = -.3$, then a one-unit change in x is associated with a $\exp(-.3) - 1 \approx .26$ or 26% decrease. Again, the approximation was biased downward relative to the exact implied percentage change in y.