1.010 - Fall 1999
SM Practice Problems – Solution

Problem 1:

The difficulty of the problem consists in resisting the temptation of working on Y_1 and Y_2 separately (or working with $\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$), because one would need to compute Cov[Y_1, Y_2] to finally get $\sigma^2_{Y_1-Y_2}$. While this method is feasible, it wastes a big chunk of time and leads to several traps (like writing Cov[Y_1, Y_2] = 0 or Cov[Y_1, Y_2] = $\rho\sigma_{Y_1}\sigma_{Y_2}$ where ρ is not known).

Recall Y_1 and Y_2:

$$Y_1 = \frac{D}{L}X_1 + \left(1 - \frac{D}{L}\right)X_2$$
$$Y_2 = \left(1 - \frac{D}{L}\right)X_1 + \frac{D}{L}X_2$$

where X_1 and X_2 are iid with mean value m and variance σ^2, and $D \leq L/2$.

We calculate $Y_1 - Y_2$:

$$Y_1 - Y_2 = \frac{D}{L}X_1 + \left(1 - \frac{D}{L}\right)X_2 - \left(1 - \frac{D}{L}\right)X_1 - \frac{D}{L}X_2$$
$$= \left[\frac{D}{L} - \left(1 - \frac{D}{L}\right)\right]X_1 + \left[\left(1 - \frac{D}{L}\right) - \frac{D}{L}\right]X_2$$
$$= \left(\frac{2D}{L} - 1\right)X_1 + \left(1 - \frac{2D}{L}\right)X_2$$

Method 1: (quick formulas for $\underline{X}$ having 2 components)

Using formulas of second moment analysis:

$$m_{Y_1-Y_2} = a_0 + a_1 m_{X_1} + a_2 m_{X_2} = 0 + \left(\frac{2D}{L} - 1\right)m + \left(1 - \frac{2D}{L}\right)m$$

$$\boxed{m_{Y_1-Y_2} = 0}$$

$$\sigma^2_{Y_1-Y_2} = a_1^2\sigma^2_{X_1} + a_2^2\sigma^2_{X_2} = \left(\frac{2D}{L}-1\right)^2\sigma^2 + \left(1-\frac{2D}{L}\right)^2\sigma^2$$

$$\boxed{\sigma^2_{Y_1-Y_2} = 2\left(1-\frac{2D}{L}\right)^2\sigma^2}$$

<u>Method 2:</u> (matrix notations)

Let $\underline{\underline{B}} = \begin{bmatrix} \frac{2D}{L}-1 & 1-\frac{2D}{L} \end{bmatrix}$ and $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$, then:

$Y_1 - Y_2 = \underline{\underline{B}}\ \underline{X}$

Using second-moment analysis:

$$m_{Y_1-Y_2} = \underline{\underline{B}}\ \underline{m}_{\underline{X}} = \begin{bmatrix} \frac{2D}{L}-1 & 1-\frac{2D}{L} \end{bmatrix}\begin{bmatrix} m \\ m \end{bmatrix} = \left(\frac{2D}{L}-1\right)m + \left(1-\frac{2D}{L}\right)m$$

$$\boxed{m_{Y_1-Y_2} = 0}$$

$$\sigma^2_{Y_1-Y_2} = \underline{\underline{B}}\ \underline{\underline{\Sigma}}_{\underline{X}}\ \underline{\underline{B}}^T = \begin{bmatrix} \frac{2D}{L}-1 & 1-\frac{2D}{L} \end{bmatrix}\begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}\begin{bmatrix} \frac{2D}{L}-1 \\ 1-\frac{2D}{L} \end{bmatrix}$$

$$= \sigma^2\left(1-\frac{2D}{L}\right)^2 \begin{bmatrix} 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\boxed{\sigma^2_{Y_1-Y_2} = 2\left(1-\frac{2D}{L}\right)^2\sigma^2}$$

Differential settlement is the difference between the settlements of each pile of the beam. We calculated that $m_{Y_1-Y_2} = 0$. This does not mean there is no differential settlement. Differential settlement exists because $Y_1 - Y_2$ is a random variable (i.e. 0 in average, yet not always equal to 0). Differential settlement is more of a problem when there is some possible "large error" for $Y_1 - Y_2$. In other words, the differential settlement is more of a problem when the variance is large.

$$\sigma^2_{Y_1-Y_2} = 2\left(1-\frac{2D}{L}\right)^2\sigma^2$$

For D varies from 0 to L/2, $\sigma^2_{Y_1-Y_2}$ decreases from $2\sigma^2$ to 0. It is maximum for D = 0. In the case D = L/2, the load is concentrated in the middle and $Y_1 = Y_2$ independently of the values of X_1 and X_2. This explains why the variance of the difference is 0.

Therefore, the differential settlement is more of a problem for small values of D.

Problem 2:

Taylor expansion around the mean of $X \sim N(m_X, \sigma_X^2)$ gives:

$$Y = g(X) \approx g(m_X) + g'(m_X)(X - m_X)$$
$$= e^{m_X} + e^{m_X}(X - m_X)$$

for which, we have:

$$\boxed{m_Y^{approx} = g(m_X) = e^{m_X}} \quad \text{-vs-} \quad m_Y = e^{m_X + 0.5\sigma_X^2}$$

$$\boxed{\sigma_Y^{2\ approx} = [g'(m_X)]^2\sigma_X^2 = e^{2m_X}\sigma_X^2} \quad \text{-vs-} \quad \sigma_Y^2 = m_Y^2(e^{\sigma_X^2} - 1) = e^{2m_X + \sigma_X^2}(e^{\sigma_X^2} - 1)$$

$$\boxed{V_Y^{approx} = \frac{\sigma_Y}{m_Y} = \frac{e^{m_X}\sigma_X}{e^{m_X}} = \sigma_X} \quad \text{-vs-} \quad V_Y = \frac{\sigma_Y}{m_Y} = \sqrt{e^{\sigma_X^2} - 1}$$

From these results, one can see that the approximated mean, variance, and coefficient of variation of Y are close to the true values if $\sigma_X^2 << m_X$ and $\sigma_X^2 << 1$. Indeed, for $\sigma_X^2 << 1$, one can write that $e^{\sigma_X^2} - 1 \approx \sigma_X^2$ (recall that for u small, $e^u \approx 1 + u$). Thus, the variance has to be very small.
For a normal distribution (which is the case of X), these conditions mean that the distribution of X is localized around its mean. Therefore, f_X has a very "narrow profile". In the "narrow" interval of X, e^X can be approximated as linear, and the FOSM is accurate.

Note 1: explaining that g(X) has to be linear for the results to be accurate is not sufficient. e^X is obviously non linear, but there are conditions for which the FOSM analysis on e^X is accurate.
Note 2: writing that X has to be around its mean for the results to be accurate is true but not precise enough, because this statement is always true in general for g non linear.

Note 3: writing that the variance of X has to be exactly equal to 0 is not precise either. In that case, X is reduced to one value (its mean). Linearization for one point makes no sense.

<u>Problem 3:</u>

(a) We calculate the mean value and variance of Y_1. Recall that:

$$m_Y = \sum_{\text{all } y_i} y_i P_Y(y_i) \qquad \text{and} \qquad \sigma_Y^2 = \sum_{\text{all } y_i} (y_i - m_i)^2 P_Y(y_i)$$

Therefore,

$$m_{Y_1} = 0 \times P_{Y_1}(0) + 1 \times P_{Y_1}(1) = 0 \times [P_{Y_1,Y_2}(0,0) + P_{Y_1,Y_2}(0,1)] + 1 \times [P_{Y_1,Y_2}(1,0) + P_{Y_1,Y_2}(1,1)]$$
$$= 0 \times (0.5 + 0.1) + 1 \times (0.1 + 0.3)$$

$$\boxed{m_{Y_1} = 0.4}$$

$$\sigma_{Y_1}^2 = (0 - 0.4)^2 \times P_{Y_1}(0) + (1 - 0.4)^2 \times P_{Y_1}(1) = 0.4^2 \times (0.5 + 0.1) + 0.6^2 \times (0.1 + 0.3)$$

$$\boxed{\sigma_{Y_1}^2 = 0.24}$$

Similar calculations on Y_2 give identical results.

(b) We calculate the covariance of Y_1 and Y_2. Recall that:

$$\text{Cov}[Y_1, Y_2] = E[(Y_1 - m_{Y_1})(Y_2 - m_{Y_2})]$$

Therefore:

$$\text{Cov}[Y_1, Y_2] = (0 - m_{Y_1})(0 - m_{Y_2})\ P_{Y_1,Y_2}(0,0) + (0 - m_{Y_1})(1 - m_{Y_2})\ P_{Y_1,Y_2}(0,1)$$
$$+ (1 - m_{Y_1})(0 - m_{Y_2})\ P_{Y_1,Y_2}(1,0) + (1 - m_{Y_1})(1 - m_{Y_2})\ P_{Y_1,Y_2}(1,1)$$
$$= 0.4^2 \times 0.5 - 2 \times 0.4 \times 0.6 \times 0.1 + 0.6^2 \times 0.3$$

$$\boxed{\text{Cov}[Y_1, Y_2] = 0.14}$$

The covariance is positive. This is expected because the outcome $Y_1 = 0$ is associated with a larger probability of occurrence of $Y_2 = 0$. Similarly, the outcome $Y_1 = 1$ is associated with a larger probability of occurrence of $Y_2 = 1$. A negative covariance would mean that $(Y_1, Y_2) = (0, 1)$ or $(Y_1, Y_2) = (1, 0)$ are more common occurrences than $(0, 0)$ or $(1, 1)$.