# 8

# Stresses/Deflections Shafts in Torsion

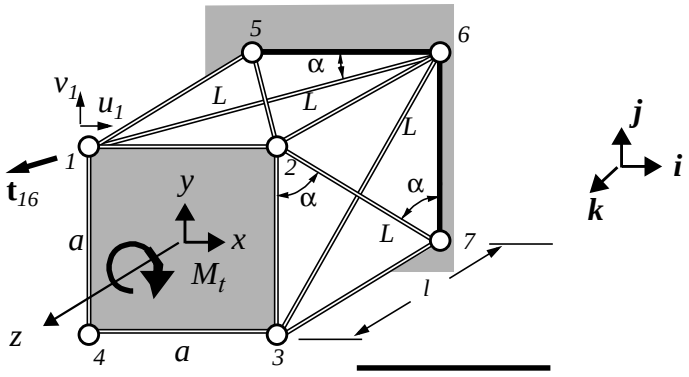
## 8.1 An Introductory Exercise

We return to the problem of torsion of circular shafts. We want to develop methods to determine the *shear stress distribution over the cross-section* of the *torque-bearing structural element* and the r*otation of any cross-section* relative to another. Although we limit our attention to circular cross-sections, this ought not to be taken to imply that only circular shafts are available to carry *torsional loads*. For example:

**Exercise 8.1**

*A single bay of a truss structure, typical of the boom of a construction crane, is shown below. Show that the torsional stiffness of the section 1-2-3-4 relative to the other, fixed section, is given by*

$$M_T = K_T \cdot \phi \qquad where \qquad K_T = 2(AEa) \cdot (cos\alpha)^3$$



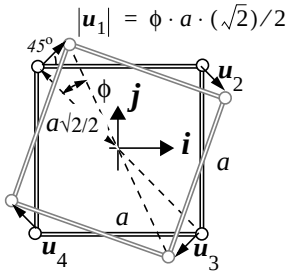

In this, $\cos\alpha = a/\sqrt{(a^2+l^2)}$ and $\phi$ is the rotation of the section 1-2-3-4.

To determine a *stiffness* we must necessarily consider displacements of the structure at the four, *unconstrained* nodes. But since we are only interested in the *torsional stiffness*, that is, the relation between a torque applied at the free end section and the rotation of that same section, our task is lighter.

Our method of analysis will be a *displacement formulation*. That is, we will impose a *displacement field* of a particular kind, namely a *rigid body rotation* of

the plane section 1-2-3-4 about the $z$ axis of the structure and determine what forces are required to be applied at the nodes to maintain the displaced configuration. Rigid here means that none of the truss members lying in the plane of this section experience any change in length; their end nodes retain their same **relative** positions - the square section remains a square.

The figure shows the imposed rotation about the $z$ axis; the rotation $\phi$ is understood to be *small* in the sense that the magnitude of the displacements of the nodes can be taken as the product of the half-diagonal of the section and the angle. Vector expressions for the displacements are then:




$$\mathbf{u}_1 = (a/2)\phi\mathbf{i} + (a/2)\phi\mathbf{j} \qquad \mathbf{u}_2 = (a/2)\phi\mathbf{i} - (a/2)\phi\mathbf{j}$$

$$\mathbf{u}_3 = -(a/2)\phi\mathbf{i} - (a/2)\phi\mathbf{j} \qquad \mathbf{u}_4 = -(a/2)\phi\mathbf{i} + (a/2)\phi\mathbf{j}$$

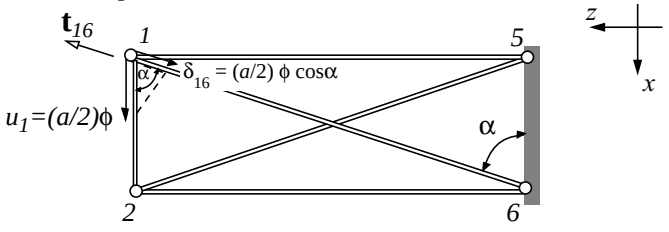
Now if the truss members lying in the plane of the section do not deform, they do not offer any resistance to rotation. But then what members **do** resist this particular *displacement field* and provide the torsional stiffness?

We inspect them all. Consider member 1-5: If we do not allow a $z$ displacement, its change in length will be zero. It is zero because, even though node #1 displaces relative to node #5, the displacement vector of #1 is perpendicular to the member; the projection of the displacement upon the member is zero. The same can be said about members 2-6, 3-7, and 4-8.

The same **cannot** be said of member 1-6, or any of the other diagonal members. The displacement of node #1 **does** have a non-zero projection upon the member 1-6, a projection tending to decrease its length when $\phi$ is positive as shown. To determine the magnitude of the contraction (as well as certifying that indeed the member does contract) we proceed formally, constructing a unit vector along the member in the direction *outward* from node #1, then determine the projection by taking the scalar product of the displacement and this unit vector. The unit vector is $\mathbf{t}_{16} = -\cos\alpha \cdot \mathbf{i} + \sin\alpha \cdot \mathbf{k}$ . The change in length of the member is then

$$\delta_{16} = \mathbf{t}_{16} \bullet \mathbf{u}_1 = -(a/2)\phi\cos\alpha$$

Less formally, we can try to visualize how the displacement **does** produce a contraction. We show a top view:



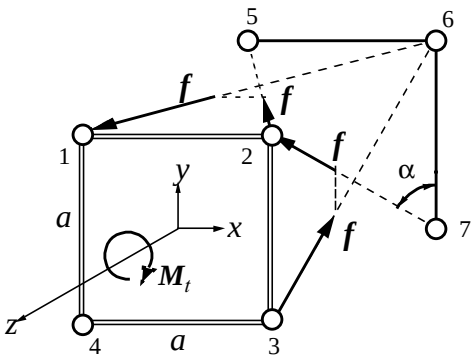

We note that, of the two scalar components of $\boldsymbol{u}_1$, the y component, $(a/2)\phi\,\boldsymbol{j}$ - $\boldsymbol{j}$ out of paper - will be perpendicular to the member and hence it will not change

the member's length in any way. The same cannot be said for the other component, the $x$ component $(a/2)\phi\,\boldsymbol{i}$.

Hopefully it is clear that the projection of this component along the member is just $(a/2)\phi\cos\alpha$ as obtained from the dot product. Note that it acts to shorten member 1-6. Similarly, the other diagonal members, 2-7, 3-8, and 4-5 will also contract and by the same amount.

Using a similar argument, we can show that members 2-5, 3-6, 4-7 and 1-8 will **extend** the same amount.

Compressive and tensile forces will then be engendered in the diagonals by this particular displacement field and it is these which resist the applied torque.




To see how, we make an isolation of the sections and require moment equilibrium about the $z$ axis.

First note that force equilibrium in the three coordinate directions $x$, $y$, *and* $z$ is satisfied: Since all eight diagonal members experience the same magnitude of extension or contraction the tensile or compressive force each will experience will be of the same magnitude. So at each node, the $z$ component of one in tension will just balance the $z$ component of the another in compression.

We denote the value of the tensile load by $f$. Assuming a linear, elastic force-deformation relationship for all diagonal members of cross-sectional area $A$ and elastic modulus $E$, we can write:

$$f = (AE/L)\cdot\delta$$

The magnitude of the extension or contraction of each diagonal is the same

$$\delta = \delta_{16} = (a/2)\cdot\phi\cdot\cos\alpha$$

$$\text{so} \qquad f = \left(\frac{AE}{L}\right)\cdot\delta = \left(\frac{AE}{L}\right)\cdot(a/2)\cdot\phi\cdot\cos\alpha = (AE/2)\cdot\phi\cdot\cos\alpha^2$$

where we have used the fact that $\displaystyle L = \frac{a}{\cos\alpha}$

Now **each** of the diagonal members produces a moment about the $z$ axis. More precisely, the **horizontal** components of the member forces in the top and bottom diagonals and the **vertical** components of the member forces in the diagonals along the sides each contributes a moment of magnitude $f\cos\alpha\ (a/2)$. All eight together then provide a torque resisting the imposed rotation $\phi$ of

$$\boxed{M_T = 4af\cdot\cos\alpha = 2aAE(\cos\alpha)^3\cdot\phi}$$

Here, then, is a structure capable of resisting torsion about its axis. Observe that:

- The units check. *AE* has the units of force since *E* the elastic modulus, has the units of stress or force per unit area: *a* has the dimensions of length so the right-hand side has the dimensions of the left, that of force times length, that of a moment or torque.
- As we increase the length of the bay relative to the length of the side of the square cross-section, the angle $\alpha$ increases, $\cos\alpha$ decreases, and the torsional stiffness decreases - and dramatically since *it goes as the cube of the cosine*.
- The relationship between torque and rotation $\phi$ is *linear* because we assumed *small displacements and rotations*.
- Imagine a structure built up of many of these bays fastened together. If this structure includes ten bays aligned along the *z* axis, then the torsional stiffness would be one-tenth the value obtained for one bay alone since the over-all rotation would be the sum of ten relative rotations of the same magnitude. The torque acting on each bay is the same. We might say the total angular rotation is *uniformly distributed* over the full string of ten bays.
- We can go back and determine the forces and/or normal stresses in the diagonal members given an applied torque $M_T$. We have:

$$M_T = 4af \cdot \cos\alpha \qquad \text{so} \qquad f = M_T/(4a\cos\alpha)$$

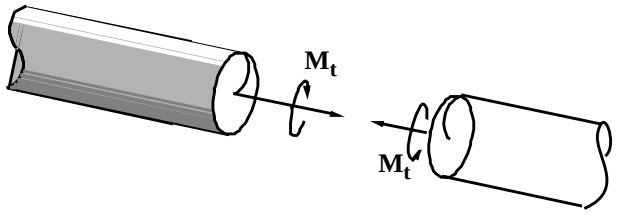
- Finally, the structure is redundant. We could remove some members and still carry the load. But note that if we remove one of diagonal members from each of the four sides of the bay our equilibrium requirement in the *z* direction would not be satisfied by the force system posed. We would have to apply a force in the *z* direction at each node in order to maintain our prescribed displacement field. Without this additional constraint, the nodes would displace in the z direction and in an un-rotationally symmetric fashion. Our analysis would not go through.

In analyzing the torsion of a circular shaft we will proceed much the same way as above. We will first consider deformations due to a relative rotation of two sections of the shaft and, on the basis of symmetry, construct a compatible strain state. The stress-strain equations give a corresponding stress distribution— one which consists solely of a shear stress acting in the plane of the cross-section. Equilibrium, repeating the maneuvers of a previous chapter, then brings the applied torque into the picture and we end with an equation relating the applied torque to the rotation of the shaft, a *stiffness relation* in form like the one derived above for the truss bay.

## 8.2 Compatibility of Deformation

The cross-sections of a circular shaft in torsion rotate as if they were rigid in-plane. That is, there is no relative displacement of any two, arbitrarily chosen points of a cross section when the shaft is subjected to a torque about its longitudinal, $z$, axis. We prove this assertion relying on *rotational symmetry* and upon the constancy of the internal torque as we move down the axis of the shaft.

We first show that radial lines must remain straight by posing that they deform, then show a contradiction results if we do so.




Let the points along a radius take the shape of a curve in plane in the deformed state: Now consider the same set of points but from the perspective of someone who has the portion and the shaft to the right to observe. He or she would necessarily see the deformed locus as shown if the same moment is to have the same effect. Now, however, the two cannot be put back together without leaving a hole within the interior. The two displacement fields are incompatible.

The only displacement pattern that **will** fit back together is if the locus is again a straight line.

It is still possible that, while radial lines remain radial, there could be some sort of accordion effect as we march around the axis of the shaft – some radial lines coming closer together, others widening the angle between them. But no, this is not possible since we have complete rotational symmetry. Whatever happens at one angular position must happen at every other angular position.

There remains the possibility of *out of plane* bulging-out and/or dishing-in. While these would not violate rotational symmetry, we rule them out using the following argument based upon the fact that the torque does not vary as we move along the axis of the shaft.

We posit a bulging-out on a section of shaft when we apply a clock-wise moment. Running around to the other end of the section, we would claim a bulging-out there too, since the moment is again directed along the axis of the section in the same sense. But now consider the portion of the shaft to either side of the cut section. To be consistent, their cross-sections will bulge out. This is clearly an incompatible state of deformation. On the other hand, if it dished-in, then we would have a torque, the same torque producing two dramatically different effects.

This argument also rules out a uniform extension or contraction of the cross-section.

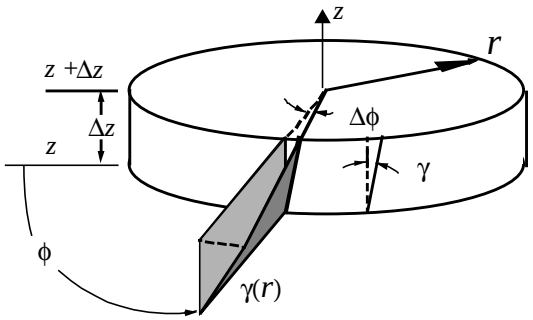
Thus, no bulging-out, no dishing-in, radial lines remain radial - a cross-section rotates as a rigid plane.

One further fact follows from the uniformity of torque at each section, namely, **the relative rotation of two cross-sections is the same for any two sections separated by the same distance along the axis of the shaft**.

If we let $\phi$ be the rotation of any section, then this is equivalent to saying

*$d\phi/dz$ is a constant*

Consider now the strains due to the rotation of one section relative to another.




The figure shows the rotation of a section located along the axis at $z+\Delta z$ relative to a section at $z$ just below it. Of course the section at $z$ has rotated too, most likely. But it is the relative rotation of the two sections which gives rise to a strain, a shear strain. $\gamma$, which measures the decrease in right angle, originally formed by two line segments, one circumferential, the other axially directed as shown. From the geometry we can state:

$$\gamma = \lim_{\Delta z \to 0} (r\Delta\phi/\Delta z) = r\frac{d\phi}{dz}$$

Note that this relationship shows that the shear strain is a linear function of radius - zero at the axis, maximum at the shaft's outer radius. Note too that, with $d\phi/dz$ a constant, the shear strain does not vary with $z$ with position along the axis of the shaft.

There are no other strains! With no deformation in plane and no bulging-out or dishing in, there are no other strains. If there existed some asymmetry like that of the truss bay structure with one diagonal number removed from each bay, then we would not be able to rule out a contraction (or extension) in the $z$ direction.

## 8.3 Constitutive Relations

Because we have but one strain component, this will be a very short section. The corresponding stress is the shear stress $\tau$ and is related to the shear strain according to:

$$\tau = G\gamma = G \cdot r\left(\frac{d\phi}{dz}\right)$$

## 8.4 The Torque-Rotation Stiffness Equation

Back in Chapter 3, we obtained an expression for resultant torque about the axis of a circular shaft due to a shear stress distribution $\tau(r)$, an arbitrary function of $r$.

We obtained. $$M_T = 2\pi\int_0^R \tau(r) r^2 dr$$

Now, with our linear function of $r$, we can carry out the integration. Doing so we obtain a relationship between the applied torque $M_T$ and the **rate of rotation** $d\phi/dz$, a *stiffness* relation. $M_T = 2\pi\int_0^R G\frac{d\varphi}{dz} r^3 dr$ or, since $d\phi/dz$ and $G$ are constants, we are left with the integral of $r^3$ and can write

$$\boxed{M_T = GJ\left(\frac{d\phi}{dz}\right)}$$

where $J$ is a function only of the geometry of the cross-section - its radius $R$. You may have encountered it as the *polar moment of inertia* $J = \int_{Area} r^2 dArea$ For the solid circular shaft $J = \pi R^4/2$.

This stiffness equation is analogous to the stiffness relationship derived for one bay of the truss structure considered at the outset of the chapter. For a shaft of length $L$, the rotation of one end relative to the other is just the integral of the **constant** rate of rotation over the length, that is, just the product of the two. We obtain then:

$$\boxed{M_T = (GJ/L)\cdot\phi}$$

This is our major result. Observe

- We can obtain the shear stress and strain distribution in terms of the applied moment by substitution. We obtain

$$\boxed{\begin{aligned}\tau(r) &= r\cdot(M_T/J)\\ \gamma(r) &= r\cdot M_T/(GJ)\end{aligned}}$$

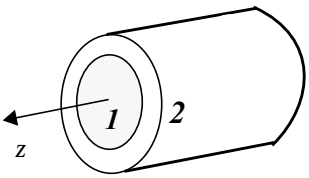
- Our analysis is identical for a hollow shaft. All of the symmetry arguments apply. Only the expression for $J$ changes: It becomes

$$J = \int_{Area} r^2 dA = (\pi/2)(R_o^4 - R_i^4)$$

where $R_0$ is the outer radius and $R_i$ the inner radius of the shaft.

- If we do anything to destroy the rotational symmetry, all bets are off. In particular if we slit a hollow tube lengthwise we dramatically decrease the torsional stiffness of the tube.

If we have a *composite shaft* of two *concentric shafts,* or more, the analysis will go through as follows:




The symmetry arguments still apply; the strain as a function of radius remains linear and proportional to the rate of twist $\gamma = rd\phi/dz$

But now our shear stress distribution is no longer the same. Within region 1, the core shaft we have $\tau_1 = G_1 \cdot r \cdot \frac{d\phi}{dz}$ and $\tau_2 = G_2 \cdot r \cdot \frac{d\phi}{dz}$ within region 2, the concentric outer shaft.

The equivalent moment is then $M_T = \int_{A_1} r\tau_1 dA_1 + \int_{A_2} r\tau_2 dA_2$ which yields

$$M_T = [G_1 J_1 + G\ J_2]\left(\frac{d\phi}{dz}\right) = (\overline{GJ})\left(\frac{d\phi}{dz}\right)$$

The shear strain distribution is then $\gamma = rM_T(\overline{GJ})$ with the shear stress distribution within each region is given by

$$\tau_1 = [G_1/(\overline{GJ})]rM_T \qquad \tau_2 = [G_2/(\overline{GJ})]rM_T$$

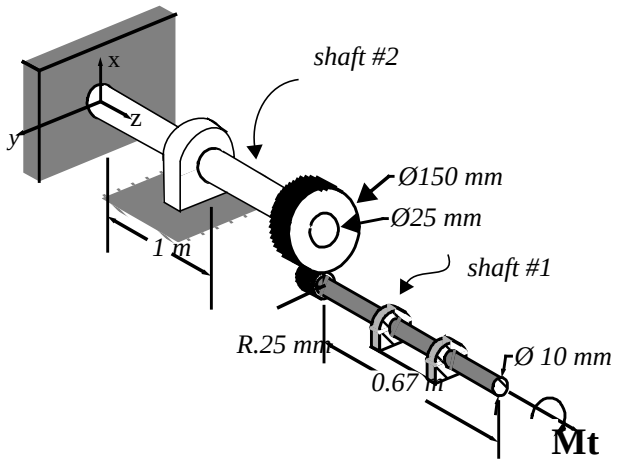
Note that, although $R_2 > R_1$ the maximum shear stress might occur at the outermost radius of the inner shaft if $G_1 > G_2$.

**Exercise 8.2**

*A torque $M_T$ of 20 Nm is applied to the steel shafts geared together as shown. Now show that*

- The internal force acting between the gear teeth is 800. *N.*
- The maximum shear stress due to torsion is $204 MN/m^2$ and occurs in the smaller diameter shaft at its outer radius.

- The torsional stiffness, $K\phi$, at the end where $M_T$ is applied is 44.35 *Nm/radian*..




Reid: Hey Katie, you do the first part and I'll do the second, ok?

Katie: Well, I don't know; usually when the author puts out a sequence of questions like this it's best to go through step by step.

Reid: That's only for the do-do's who need their hand held. See, I got the second question wired. It goes like this: the torque is the same in both, right? And the radius of the shaft that's fixed to the wall is biggest and because the maximum shear stress in a shaft is at the biggest radius, I just plug into the formula for shear stress and I've got it.

Katie: I think you're wrong Reid, I think we should go slower, take things step by step.

Reid: You're just jealous because you girls can't see things as quick as us guys.

Katie: It appears that you're so quick you didn't even need to read the problem ..... it says right there that the maximum shear stress occurs in the smaller diameter shaft!

Reid: Maybe the answer given is wrong. Stuff in these books are wrong lots of time. I mean look at that equation for shear stress

$$\tau = rM_T/J.$$

See that little old radius there up top? Now when that *r* maxes out so does $\tau$.

Katie: But Reid, that *little old J* depends on r too. In fact, for a solid shaft it goes as the fourth power to the radius so it predominates.

Reid: *Predominates*.... You're saying the smaller radius shaft has maximum shear stress for the same moment?

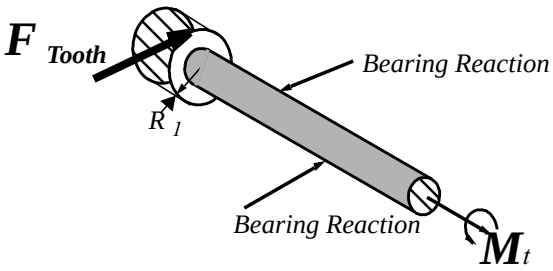
Katie: Hold on, hold on. The maximum shear stress depends upon the moment in the shaft as well.. that "little old" $M_T$ right there. So we have got to first see which shaft carries the largest torque, and then...

Reid: But the torques are the same in both shafts!

Katie: ...same in both shafts. Where do you get that from?

Reid: I dunno...conservation of torque, I guess...it just feels right.

Katie: You men are all alike. What have feelings have to do with it? Watch me: I isolate the shaft carrying the torque $M_T$, just like the author did with that interesting historical example of a human powered, well-water lift. See, I draw this figure:




Now moment equilibrium about the axis of the shaft gives me that force on the tooth is going to be

$F_{Tooth} = M_T/R_1$ where $R_1$ is the radius of the small gear.

Reid: So.....

Katie: Well, **then** I turn to the other shaft and find that the torque is $M_2 = R_2F_{Tooth} = (R_2/R_1)M_T$, which, my friend, is not equal to, but **three times** the applied torque, $M_T$. So, Reid, you see how your feelings can lead you astray?

****

Katie is correct, as is the book. The torque in the shaft fixed to the wall is three times the torque carried by the free-ended shaft. Even so, because the shear stress is more sensitive to changes in the radius of a shaft than to torque, it is the smaller shaft that sees the largest shear stress. If both the shafts had the same radius, the one fixed to the wall would indeed show the largest shear but that is not the case here. I could make up another story about how, with less of a radius, the smaller cross-section must experience a greater stress at each and every position $r$ in order to sum to the same moment.

The work follows:

The maximum stress in shaft #2 is $\tau_2|_{max} = r_2M_2/J_2$ which, with $J=\pi r_2^4/2$ gives

$\tau_2|_{max} = 39.2\ MN/m^2$

while for shaft #1 we obtain $\tau_1|_{max} = 204\ MN/m^2$

The torsional stiffness at the end where the movement is applied is obtained by summing up the relative angular deflections of the two shafts being careful to assure compatibility of deformation at the section where the gears mesh. Working from the wall out, the angular rotation of the geared end of shaft #2 relative to its (fixed) end is $\phi_2(L_2) = (M_2L_2)/(GJ_2)$

Now, compatibility of deformation, no-slip at the gear teeth requires that

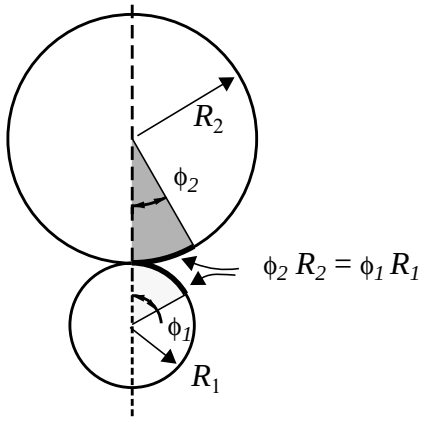
$$\phi_2(L_2)R_2 = \phi_1(0)R_1$$

where the directions of the angular rotations are indicated in the figure. So we have

$\phi_1(0)=(R_2/R_1)(M_2L_2)/(GJ_2)$

Now the **relative rotation** of the **free end** of the small shaft to which the torque is applied is given by

$\phi_1(L_1) - \phi_1(0) = (M_1L_1)/(GJ_1)$




So

$\phi_1(L_1) = (M_1L_1)/(GJ_1) + (R_2/R_1)(M_2L_2)/(GJ_2)$

Substituting for $M_1$, $M_2$ we can write

$$\phi_1(L_1)=(M_1L_1)/(GJ_1)[1 + (R_2/R_1)^2(L_2/L_1)(J_1/J_2)]$$

So $K\phi = (GJ_1/L_1)/[1 + (R_2/R_1)^2(L_2/L_1)(J_1/J_2)]$. which gives $K\phi = 44.35\ NM/radian$

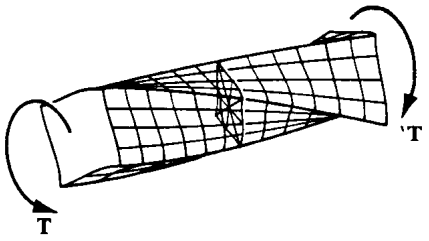
Observe:

- Although only numerical results were required to be verified, I worked through the problem symbolically refraining from *plugging-in* until the end. This is highly recommended practice for it allows me to keep a check on my work by inspecting the dimensions of the results I obtain.
- Note, too, how I can readily extend my results to other configurations - changing the relative length of the shafts, their radii, and the material out of which they are made.
- The maximum shear stress is quite large relative to the yield stress of steel. We shall see that yielding becomes a possibility when the maximum shear stress is one-half the yield strength in a tension test.

## 8.5 Torsion of other than circular shafts

We have already stated how, if a shaft does not exhibit symmetry, our analysis of deformation does not apply. A slit hollow tube, for example, behaves dramatically different from a continuous hollow tube.

Considerable effort has been expended by applied mathematicians over a century or two in developing solutions for the torsion of shafts of other than circular cross section. The results for a shaft sporting a rectangular cross section are well documented[1]. The figure at the right is a classic.




The distorted shape – note the *warping* of the originally plane cross section – is due, in part, to the fact that the shear stresses at the corners must vanish.

Other results say that the maximum shear stress occurs in the middle of the longest side of length *b*. Its magnitude and that of the torsional stiffness are neatly summarized below. $\tau|_{max} = M_t/[k_2(2a)^2(2b)]$

is the maximum shear stress. And $M_t = K_t\phi/L$ *where* $K_t = k_1 G(2a)^3(2b)$

is the torsional stiffness. Note: $a$ and $b$ are the dimensions of the cross-section. Values for the constants $k_1$ *and* $k_2$ as functions of the ratio of the lengths of the sides of the cross section are given in the table below, again taken from Timoshenko and Goodier.

TABLE OF CONSTANTS FOR TORSION OF A RECTANGULAR BAR

| $\frac{b}{a}$ | $k$ | $k_1$ | $k_2$ | $\frac{b}{a}$ | $k$ | $k_1$ | $k_2$ |
|---|---|---|---|---|---|---|---|
| 1.0 | 0.675 | 0.1406 | 0.208 | 3 | 0.985 | 0.263 | 0.267 |
| 1.2 | 0.759 | 0.166 | 0.219 | 4 | 0.997 | 0.281 | 0.282 |
| 1.5 | 0.848 | 0.196 | 0.231 | 5 | 0.999 | 0.291 | 0.291 |
| 2.0 | 0.930 | 0.229 | 0.246 | 10 | 1.000 | 0.312 | 0.312 |
| 2.5 | 0.968 | 0.249 | 0.258 | ∞ | 1.000 | 0.333 | 0.333 |

Reprinted with permission from S.P. Timoshenko and J.N. Goodier, *Theory of Elasticity,* (1970, McGraw-Hill, Inc.)

1. The results presented here are drawn from *the* classic text about stress and strain – Timoshenko and Goodier, *THEORY OF ELASTICIY,* Third Edition, McGraw-Hill, 1970.

**Design Exercise 8.1**

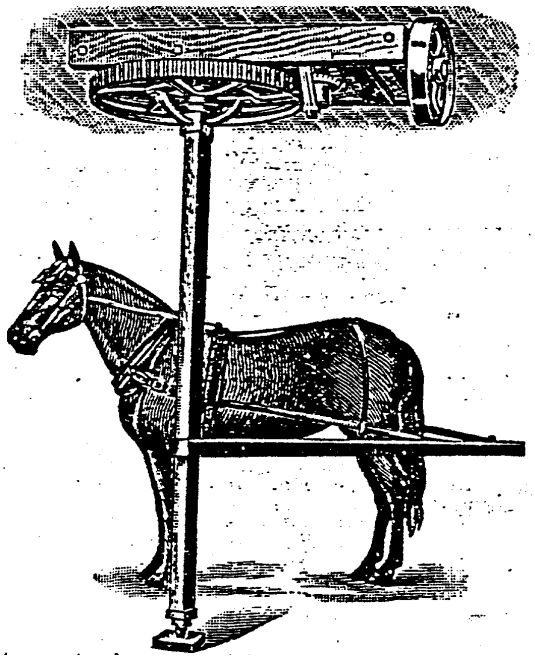
A hollow aluminum shaft, two meters long, must transmit a torque of 20 *KNm*. The total angle of twist over the full length of the shaft is not to exceed $2.0^{o}$, and of course, we do not want the shaft to yield. Size the outside and inside diameters of the shaft.

## 8.6 Problems – Torsion of Circular Shafts

**8.1** *What if* a solid circular shaft is replaced by a square shaft whose **diagonal** is equal to the diameter of the original circular shaft; How does the torsional stiffness change; For the same torque, how does the maximum shear stress change?

*What if* the solid circular shaft is replaced by a square shaft whose **side** is equal to the diameter of the original circular shaft; How do these change?

**8.2** *Estimate* the maximum shear stress in the square, center-post shaft of the drive transmission system shown below when the horse is delivering $\alpha$ *horsepower* where $\alpha < 1.0$. (1 *horsepower* = 746 *Watts* = 550 *ft-lb*/*sec*) Estimate the maximum stress due to **bending** in the *sweep arm*. Estimate the tension in the belt (not shown) that would be placed over the *driving pulley.*



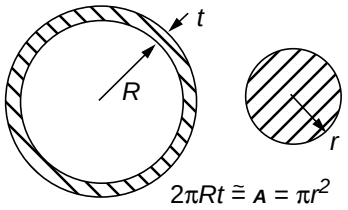
This style of horse power is very convenient and popular, because, owing to its construction, it has many advantages not found in down powers. It is especially adapted for use in a barn where several horses are kept, or in small livery stables. The power can be bolted to the timbers above the driveway and machines can be set on the floor either above or below the power. When not in use the center post can be lifted from its socket and put out of the way, leaving the floor clear for other purposes. Then when power is to used again all that is necessary is to set the post in place, hitch the horse to the sweep and go ahead. The center post which we furnish is made of 6-inch by 6-inch timber and is 12 feet long. It is amply strong and can be cut to any desired length. The 1 1/8-inch driving shaft, to which the pulley is attached, is regularly made so that the measurement from center of master wheel to center of pulley face is 3 feet 3 inches, but additional shafting can be coupled to this shaft so as to change position of pulley or allow the use of other pulleys. The driving pulley is 18 inches in diameter with 3-inch face and makes 37 1/3 revolutions to one round of the horse, or about 135 revolutions per minute. Additional shafting or change in size of pulley is extra. Length of sweep from center post to eye-bolt is 7 feet 6 inches. For driving small feed cutters, corn shellers, feed grinders, wood saws, etc., this power cannot be excelled. Weight, 450 pounds. Shipped direct from factory in Southeastern Wisconsin. No. 32R1823 .....$18.45[2]

**8.3** A thin Aluminum tube, whose wall thickness is 1 *mm*, carries a torque 80% of the torque required for the onset of yield. The radius of the tube is 20 *mm*.

*Show that* an estimation of shear stress, based upon the assumption that it is uniformly distributed over the thickness of the tube (and using an estimate of *J* that is linear in the wall thickness), gives a value within 10% of that computed using the full expressions for shear stress and polar moment of inertia.

**8.4** A solid aluminum, circular shaft has length 0.25 *m* and diameter 5 *mm*. How much does one end rotate relative to the other if a torque about the shaft axis of 10 *N-m* is applied?

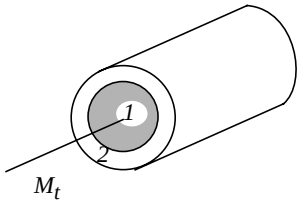
**8.5** A relatively thin walled tube and a solid circular shaft have the same cross-sectional area. You are to compare the torsional stiffness of one to the other.




4a) What does the phrase "torsional stiffness" mean?

4b) What is the ratio of the torsional stiffness of the tube to that of the solid shaft?

**8.6** A composite, cylindrical shaft has a core of one material, #1, bonding firmly to an outer, concentric, hollow shaft of another material, #2. It can be shown that the shear strain at any radial distance *r* from the center of the shaft is still given by




$$\gamma = r \cdot \frac{d\phi}{dz}$$

ie., as if it were a solid, homogeneous shaft.

Let $R_1$ be the radius of the core, $R_2$ that of the hollow shaft, take $G_2$ = G and $G_1$ = 2G respectively be the shear modulus of the two materials, construct an expression for the shear stress distribution as a function of r in terms of the applied torque and other relevant properties.

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2. 1902 Edition of The Sears, Roebuck Catalogue.