16.61 Spring 2006
Lecture 3 Lecture Notes
ver 1.0

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Updates

- Expectations: During the week I respond to e-mail withing 24 hours, but I may not be able to answer e-mail during weekends. Questions e-mailed 5p Friday through 12midnight Sunday will be answered by Tuesday 9am.

Recap

Coriolis' Theorem: Coriolis' Theorem is the main result in understanding how the time rate of change of vector quantities are related in frames that are rotating with respect to each other.

$$\left\{\frac{dr}{dt}\right\}_A = \left\{\frac{dr}{dt}\right\}_B + S(\omega)r$$

Rotation Groups: Here are some of the very basic properties of rotation matrices R.

$$\begin{aligned} R^{-1} &= R^T \\ RR^T = R^T R &= I \\ \det R &= +1 \\ \dot{R} &= S(\omega)R \\ R(t) &= e^{S(\omega t)}R(0) \end{aligned}$$

Rotation matrices and the rotation group SO(3) play a very important part in the dynamics of rigid bodies because they encode all the rules of rotational kinematics! They show up in robotics, aircraft dynamics, spacecraft dynamics, and in dynamical simulations.

Kinematic Expression for Absolute Acceleration: Here is one of the most important kinematic relations derived in this first section of the course. You should thoroughly know how apply this relation because is is then used to apply Newton's law to formulate dynamics:

$$\ddot{r}_A = \ddot{r}^{B/A} + \ddot{r}_B + \dot{\omega} \times r_B + 2\omega \times \dot{r}_B + \omega \times \omega \times r_B$$

General Approach to Newton-Euler Dynamics :

1. SETUP frames
2. WRITE the kinematic relation
3. CALCULATE qantities needed for bookkeeping
4. PLUG 'N CHUG to find INERTIAL accelerations
5. Apply Newton's principle and or Euler's principle to derive the DYNAMICS

INSIGHT:

1. Use frames to "eliminate" contraint forces
2. Use both Newton's principle and Euler's principle to get the number of dynamical equations that you need!

Today

- Examples! (You work them!)
- Tricks with cylindrical coordinates
- Frames: Homogenous Transformations
- Frames: Multiple frames
- Frames: Screw Theory

Thurday

Tentative plan:

- Numerical Coriolis examples.
- Summing up angular velocities in different frames.
- Euler's principle for a system of particles.
- More discussion on frames and "representations" of elements R of the rotation group SO(3). Perhpas Euler angles.

Problems

The problems we are working in class are to build facility using frames and to develop insight into using frames to derive Newton-Euler dynamics.

INSIGHT:

1. Use frames to "eliminate" contraint forces
2. Use both Newton's principle and Euler's principle to get the number of dynamical equations that you need!

Problem I

Simple pendulum but origin of B frame moved. Comments?

Problem II

Overhead crane. What to do about the resulting expression for absolute accleration?

Problem III

Various versions of the Merry-Go-Round problem. What do we learn about Coriolis?

INSIGHT: The Merry-Go-Round problem is kinematically equivalent to the revolute-prismatic robot! Many kinematic problmes are equivalent!

Calculations with Cylindrical Coordinates

INSIGHT: Choose the intertial frame of choice!! Calculate absolute acceleration. Then rotate to another equivalent INERTIAL frame.

A Little More on Frames: Homogenous Transformations

Homogenous Transformations

We have discovered that if two frames are related to each other by rigid motions, then they are related to each other by a translation and a rotation. In previous lecture notes, the group of all proper Euclidean motions in three dimensions has been denoted by E. This is acceсptable, but the more common mathematical notation for the group of proper Euclidean motions is $SE(3)$

Just as we had a matrix representation for proper rotations, R being a matrix represenation for and element of the group of special orthogonal rotations, SO(3), there is (at least one) matrix representation for the group of proper Eulidean motions (translations+rotations), $SE(3)$. Let's develop this notation because it has many applications in robotics, computer vision, and modern dynamics. The development is this section is purely kinematic!

A point in p frame B has the following representation in frame A

$$p_A = R_A^B p_B + r_A^B$$

Changing notation slighly for the moment we can write

$$p_A = R_A^B p_B + d_A^B$$

If we adopt the homogenous reprentation of p_A

$$\begin{pmatrix} p_A \\ 1 \end{pmatrix}$$

We can then write a homogenous transformation H representation for the rigid motion in SE(3)

$$p_A = H_A^B p_B$$

$$\begin{pmatrix} p_A \\ 1 \end{pmatrix} = \begin{pmatrix} R_A^B & d_A^B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_B \\ 1 \end{pmatrix}$$

The inverse of this transformation is given by

$$\begin{pmatrix} p_B \\ 1 \end{pmatrix} = \begin{pmatrix} (R_A^B)^T & -(R_A^B)^T d_A^B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_A \\ 1 \end{pmatrix}$$

You can verify this at home!

Multiple Frames

If we have three frames, A, B and C with $R_B = R_B^C$ and $R_A = R_A^B$, and $d_B^C = d_B$, and $d_A^B = d_A$, the we can write a composite homogenous transformation relating a point p_C in the C frame to the same point p_A expressed in the A frame.

$$p_A = H_A^B H_B^C p_C$$

$$H_A^B H_B^C = \begin{pmatrix} R_A & d_A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} R_B & d_B \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} R_A R_B & R_A d_B + d_A \\ 0 & 1 \end{pmatrix}$$

The "rule" $R_A d_B + d_A$ expresses the fact that SE(3) is a semi-direct product of $\mathbb{R}^3$ and SO(3).

$$\mathrm{SE}(3) = \mathbb{R}^3 \circledS \mathrm{SO}(3)$$

Can you symbolically form the inverse of this tranformation using the inverse relationship previously given?

We can now calculate the absolution acceleration of point p in the A frame!

$$p_A = \begin{pmatrix} R_A R_B & R_A d_B + d_A \\ 0 & 1 \end{pmatrix} p_C$$

First calculate $\dot{p}_A$ carefully applying the chain rule and using the substituton $\dot{R} = S(\omega)R$. Then calculate $\ddot{p}_A$ again carefully applying the chain rule and using the substition $\dot{R} = S(\omega)R$. Try this at home!

The matrix version of accounting may prove useful for tracking and bookkeeping calculations. Most o the Gibbs notation for multiple frames is very messy. For more than two frames, automated systems are best, HOWEVER, you should ALWAYS be able to verify computer package calculations by hand!

Ball's Screw Theory

As you might expect since

$$R(t) = e^{S(\omega)t}$$

and

$$\dot{R}(t) = S(\omega)R(t)$$

then

$$H(t) = e^{T(t)}$$

for some matrix T. Turns out this is true and that T has the following form

$$T = \begin{pmatrix} S(\omega) & v \\ 0 & 0 \end{pmatrix}$$

The $S(\omega)$ term we are familiar with from before. The v term in some sense represents a "velocity". In theory, the T matrix is said to belong to the Lie algebra $se(3)$.

It can be shown that $H(t) = e^{T(t)}$ represents a screw motion. That is a helical motion in space about some axis. Ball created his theory of screws in 1900 and it is used quite extensively today in robotics and increasingly in dynamics.

Since SE(3) has pure rotations and pure translations, there matrices H that only represent translations and rotations. These matrices may be easily composed to give an overall homogenous tranformation

$$H = H_{rotation2}H_{translation2}H_{translation1}H_{rotation1}$$

The matrices are very often used to represent the kinematics of terrestrial robots.

Sometimes $T(t)$ is written as ξt and then the composite homogenous transformation would be written as

$$H = e^{\xi_{R2}}e^{\xi_{T2}}e^{\xi_{T1}}e^{\xi_{R1}}$$

Example: Kinematics of Multiple Frames

Helicopter example! A helicopter is moving vertically with speed 2ft/s and accelerations 5 ft/s^2. At the same time, the body of the body of the helicopter is rotating about a vertical axis with a constant angular velocity of 1.4 rad/s.

The tail rotor (radius 2.5 ft) is rotating at a constant rate of 180 rad/s relative to the body. Find the absolute acceleration of a point at the tip of the tail rotor at the instant the blade is in the vertical position. The hub of the tail rotor is 25 ft behind the axis of rotation.

Can you setup this problem using the following:

$$p_A = \begin{pmatrix} R_A R_B & R_A d_B + d_A \\ 0 & 1 \end{pmatrix} p_C$$

and your derived expressionf for $\ddot{p}_A$.

Next Time

1. More Coriolis
2. More about representations of SO(3)
3. Euler's equation for systems of particles