16.61 - Aerospace Dynamics Spring 2006
Homework 3

Version: 1.0
Prof Charles P. Coleman

Date Out: Friday 7 April 2006
Date Due: Friday 14 April 2006 5pm

	Time Spent [minutes]
Problem 1	
Problem 2	
Problem 3	

Homework 3
Solution Sketch

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Problem 1

As shown in Figure 1, a mass m is attached to one end of a light rod of length l. The other end of the rod is pivoted so that the rod can swing in a plane. The pivot rotates in the same plane at angular velocity ω in a circle of radius R. Assume that gravity g acts in the negative y direction. Use Lagrangian analysis to find the equations of motion of the mass.
Hint. Take θ as the generalized coordinate. Find the position of the mass (x, y) as a function of θ. Then calculate the kinetic energy T and potential energy V of the mass.

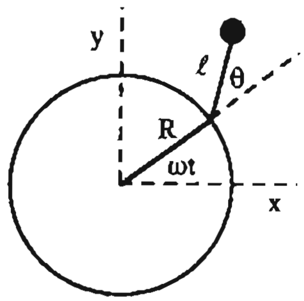


Figure 1: Pivoted Pendulum

References:
[1] Calkin, Lagrangian and Hamiltonian Mechanics, Chapter III, Problem 3, p 55, World Scientific, 1996.
[2] Calkin, Lagrangian and Hamiltonian Mechanics, Solutions to the Exercise, Chapter III, Exercise 3.03, pp 53-54, World Scientific, 1999.
[3] Greenwood, Principles of Dynamics, Chapter 6, Problem 6-6, p 276-77, Prentice-Hall, 1965.
[4] 16.61 OCW, Assignment #6, Problem 3, 2003.

Problem 2

As show in Figure 2, a spring pendulum is attached to a rotating shaft by an arm of length d. The spring stiffness is k, the pendulum mass is m, and gravity acts in the minus Z direction. The rest length of the pendulum is $r_0 = L$. The shaft is rotating with constant angular velocity Ω. Let the generalized coordinates q be $q = (\theta, r)$, where θ is the angle the pendulum makes with the vertical, and r is the distance from the pendulum pivot to the location of the mass. Use Lagrangian techniques to find the equations of motion.

References:
[1] 16.61 OCW, Assignment #3, Problem 1, 2003.
[2] 16.61 OCW, Assignment #4, Problem 1, 2003.
[3] 16.61 OCW, Assignment #8, Problem 1, 2003.

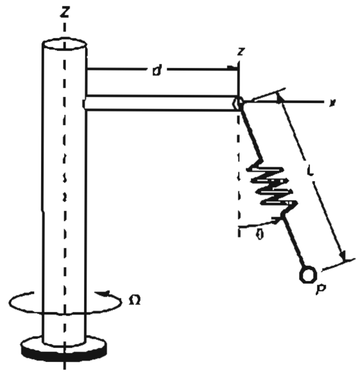


Figure 2: Swinging Pendulum

Problem 3

A sphere of mass M and radius R rolls without slipping down a triangular block of mass m that is free to move on a frictionless horizontal surface, as shown in Figure 3. The center of the sphere is initially located a distance ξ_0 from the top of the block, and the center of the sphere is initially at a height H above the horizontal surface. The sphere's moment of inertia is I. Find the Lagrangian and Lagrange's equations of motion for this system.
Hint: Take (x, θ) as the generalized coordinates. Find the position of the center of the sphere (x, y) as a function of x, ξ_0, R, θ and H. Then calculate the kinetic energy T and potential energy V of the sphere ((translational AND rotational!) and the block.

References:
[1] Lim, Problems and Solutions on Mechanics, Part II, Section 1, Problem 2022 (UC Berkeley Physics PhD Exam Question), p 506-508, World Scientific 1994.

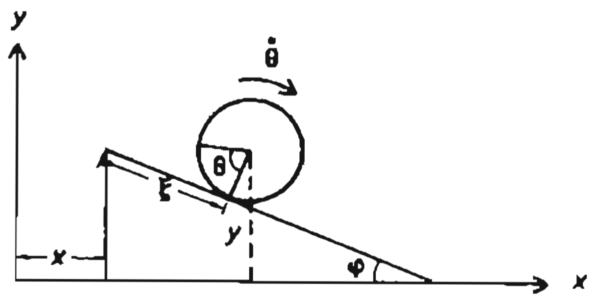


Figure 3: Sphere on Wedge

:xercise 3.02

he point of support of a simple plane pendulum moves vertically according to $y = h(t)$, 'here $h(t)$ is some given function of time.

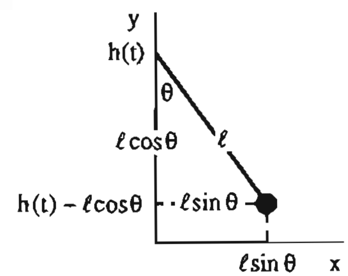
ι) Find the Lagrangian, taking as generalized coordinate the angle θ the pendulum ıakes with the vertical.

›) Write down Lagrange's equation of motion, showing in particular that the pendulum :haves like a simple pendulum in a gravitational field $g + \ddot{h}$.

olution

ı) The cartesian coordinates of the bob are (Fig. 1)

$$x = \ell \sin\theta, \qquad y = h(t) - \ell\cos\theta.$$



Ex. 3.02, Fig. 1

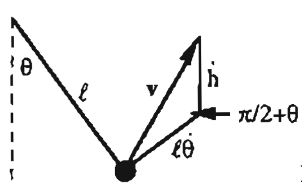
he cartesian components of the velocity of the bob are thus

$$\dot{x} = \ell\dot{\theta}\cos\theta, \qquad \dot{y} = \dot{h} + \ell\dot{\theta}\sin\theta,$$

ıd the kinetic energy is

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \tfrac{1}{2}m(\ell^2\dot{\theta}^2 + 2\dot{h}\ell\dot{\theta}\sin\theta + \dot{h}^2).$$

ıis result can instead be obtained by using the trigonometric cosine law to add the :locity $\dot{h}$ in the vertical direction to the velocity $\ell\dot{\theta}$ in the angular direction (Fig. 2).



Ex. 3.02, Fig. 2

The potential energy is

$$V = mgy = mg(h - \ell\cos\theta).$$

The Lagrangian is

$$L = T - V = \tfrac{1}{2}m(\ell^2\dot{\theta}^2 + 2\dot{h}\ell\dot{\theta}\sin\theta + \dot{h}^2) - mg(h - \ell\cos\theta).$$

(b) We have

$$\frac{\partial L}{\partial\dot{\theta}} = m(\ell^2\dot{\theta} + \dot{h}\ell\sin\theta), \qquad \frac{\partial L}{\partial\theta} = m\dot{h}\ell\dot{\theta}\cos\theta - mg\ell\sin\theta,$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = m(\ell^2\ddot{\theta} + \ddot{h}\ell\sin\theta + \dot{h}\ell\dot{\theta}\cos\theta),$$

so Lagrange's equation is

$$m\ell^2\ddot{\theta} = -m(g + \ddot{h})\ell\sin\theta.$$

This is the same as the equation of motion of a simple pendulum in a gravitational field $g + \ddot{h}$.

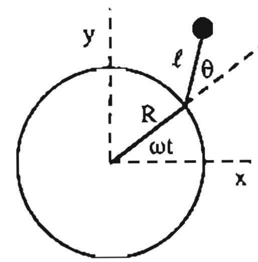
Exercise 3.03

A mass m is attached to one end of a light rod of length ℓ. The other end of the rod is pivoted so that the rod can swing in a plane. The pivot rotates in the same plane at angular velocity ω in a circle of radius R. Show that this "pendulum" behaves like a simple pendulum in a gravitational field $g = R\omega^2$ for all values of ℓ and all amplitudes of oscillation.

Solution

The cartesian coordinates of the mass are (Fig. 1)

$$x = R\cos\omega t + \ell\cos(\omega t + \theta), \qquad y = R\sin\omega t + \ell\sin(\omega t + \theta).$$



Ex. 3.03, Fig. 1

The cartesian components of the velocity of the mass are thus

$$\dot{x} = -R\omega \sin \omega t - \ell(\omega + \dot{\theta})\sin(\omega t + \theta), \qquad \dot{y} = R\omega \cos\omega t + \ell(\omega + \dot{\theta})\cos(\omega t + \theta),$$

and the kinetic energy is

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \tfrac{1}{2}m\left[R^2\omega^2 + 2R\ell\omega(\omega + \dot{\theta})\cos\theta + \ell^2(\omega + \dot{\theta})^2\right].$$

There is no potential energy, so the Lagrangian is simply T. We have

$$\frac{\partial T}{\partial \dot{\theta}} = mR\ell\omega\cos\theta + m\ell^2(\omega + \dot{\theta}), \qquad \frac{\partial T}{\partial \theta} = -mR\ell\omega(\omega + \dot{\theta})\sin\theta,$$
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = -mR\ell\omega\dot{\theta}\sin\theta + m\ell^2\ddot{\theta},$$

so Lagrange's equation is

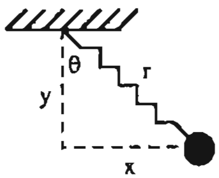
$$m\ell^2\ddot{\theta} = -mR\omega^2\ell\sin\theta.$$

This is the same as the equation of motion of a simple pendulum in a gravitational field $R\omega^2$.

Exercise 3.04

A pendulum is formed by suspending a mass m from the ceiling, using a spring of unstretched length ℓ_0 and spring constant k.
(a) Choose, and show on a diagram, appropriate generalized coordinates, assuming that the pendulum moves in a fixed vertical plane.
(b) Set up the Lagrangian using your generalized coordinates.
(c) Write down the explicit Lagrange's equations of motion for your generalized coordinates.

Solution



Ex. 3.04, Fig. 1

If we choose as generalized coordinates the cartesian coordinates (x,y) wi origin at the point of suspension and with x horizontal and y vertically down (Fig. 1), th Lagrangian is

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy - \tfrac{1}{2}k(\sqrt{x^2 + y^2} - \ell_0)^2.$$

The second term is (minus) the gravitational potential energy and the third term (minus) the potential energy due to the stretch of the spring. Lagrange's equations are

$$m\ddot{x} = -kx(1 - \ell_0/r), \qquad m\ddot{y} = mg - ky(1 - \ell_0/r),$$

with $r = \sqrt{x^2 + y^2}$.

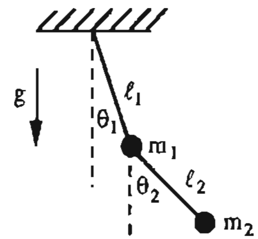
If, instead, we choose as generalized coordinates the polar coordinates (r,θ) wi r the distance of the mass from the point of suspension and θ the angle the spring mak with the vertical (Fig. 1), the Lagrangian is

$$L = \tfrac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta - \tfrac{1}{2}k(r - \ell_0)^2.$$

Lagrange's equations are then

$$m\ddot{r} = mr\dot{\theta}^2 + mg\cos\theta - k(r - \ell_0),$$
$$d(mr^2\dot{\theta})/dt = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} = -mgr\sin\theta.$$

Exercise 3.05



A double plane pendulum consists of two simple pendulums, with one pendulu suspended from the bob of the other. The "upper" pendulum has mass m_1 and length ℓ the "lower" pendulum has mass m_2 and length ℓ_2, and both pendulums move in th same vertical plane.
(a) Find the Lagrangian, using as generalized coordinates the angles θ_1 and θ_2 th pendulums make with the vertical.
(b) Write down Lagrange's equations of motion.

16.61
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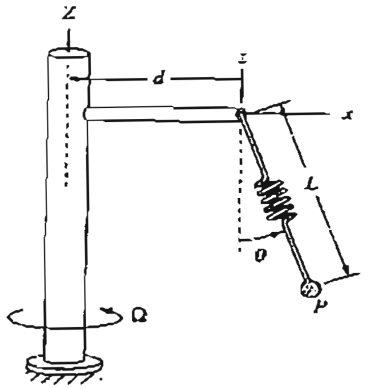
16.61 Homework Assignment #5 Solution

1. Consider the spring pendulum analyzed before. The arm attached to the rotating shaft has length d =0.8m, as shown in the figure. The shaft is rotating with a constant angular velocity of 0.4 rad/sec, but the pendulum is free to change length (L) and swing (θ). Given that the spring constant is k and the mass of point P is m, find the equations of motion for this system. Note that gravity acts.



Use Lagrange approach:

Work in the $\hat{x}-\hat{y}-\hat{z}$ frame shown in the figure
Generalized coordinates are r and θ
Constraint equation:

$$\bar{r} = (d + r\sin\theta)\hat{x} - r\cos\theta\,\hat{z}$$

Rates: $\bar{\omega} = \Omega\hat{z}$

Kinetic energy expression:

$$\dot{\bar{r}} = (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)\hat{x} - (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)\hat{z} + \Omega\hat{z}\times[(d + r\sin\theta)\hat{x} - r\cos\theta\,\hat{z}]$$

$$\dot{\bar{r}} = (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)\hat{x} + (d + r\sin\theta)\Omega\hat{y} - (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)\hat{z}$$

$$\dot{\bar{r}}\cdot\dot{\bar{r}} = (\dot{r}\sin\theta + r\dot{\theta}\cos\theta)^2 + (d + r\sin\theta)^2\Omega^2 + (\dot{r}\cos\theta - r\dot{\theta}\sin\theta)^2$$

$$v^2 = \dot{r}^2\sin^2\theta + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta + r^2\dot{\theta}^2\cos^2\theta + (d + r\sin\theta)^2\Omega^2 + (\dot{r}^2\cos^2\theta - 2r\dot{r}\dot{\theta}\sin\theta\cos\theta + r^2\dot{\theta}^2\sin^2\theta)$$

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + (d + r\sin\theta)^2\Omega^2$$

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + (d + r\sin\theta)^2\Omega^2\right)$$

Potential energy: $V = -mgr\cos\theta + \frac{1}{2}K(r - r_0)^2$

Lagrangian: $L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + (d + r\sin\theta)^2\Omega^2\right) + mgr\cos\theta - \frac{1}{2}K(r - r_0)^2$

Derivatives of r:

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 + m\sin\theta\,\Omega^2(d + r\sin\theta) + mg\cos\theta - k(r - r_0)$$

r equation of motion:

$$\ddot{r} - r(\dot{\theta}^2 + \Omega^2\sin^2\theta) + \frac{k}{m}(r - r_0) = g\cos\theta + d\Omega^2\sin\theta$$

Derivatives of θ:

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}$$

$$\frac{\partial L}{\partial \theta} = mr\cos\theta(d + r\sin\theta)\Omega^2 - mgr\sin\theta$$

$$r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - r\cos\theta(d + r\sin\theta)\Omega^2 + gr\sin\theta = 0$$

θ equation of motion:

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} - \Omega^2\cos\theta(d + r\sin\theta) + g\sin\theta = 0$$

This and Eq. (1) together give

$$x_2 = l + \frac{It}{m_1+m_2} - \frac{I\sin(\omega t)}{(m_1+m_2)\omega},$$

and thus

$$\dot{x}_2 = \frac{I}{m_1+m_2} - \frac{I\cos(\omega t)}{m_1+m_2}.$$

When m_2 comes to rest for the first time, $\dot{x}_2 = 0$, and the above gives $\cos(\omega t) = 1$ for the first time. Hence when m_2 comes to rest for the first time,

$$t = \frac{2\pi}{\omega}.$$

At that time m_2 has moved a distance

$$x_2 - l = \frac{2\pi I}{\omega(m_1+m_2)} = 2\pi I\sqrt{\frac{m_1 m_2}{k}}\,(m_1+m_2)^{-3/2}.$$

(b) If the impulse given to m_1 has a component perpendicular to the line joining the two particles the system will rotate about the center of mass, in addition to the linear motion of the center of mass. In a rotating frame with origin at the center of mass and the x-axis along the line joining the two particles, there will be (fictitious) centrifugal forces acting on the particles in addition to the restoring force of the string. At the positions of the particles where the forces are in equilibrium the particles have maximum velocities on account of energy conservation (Problem **2017**). Hence oscillations will always occur, besides the rotation of the system as a whole.

2022

A sphere of mass M and radius R rolls without slipping down a triangular block of mass m that is free to move on a frictionless horizontal surface, as shown in Fig. 2.18.

(a) Find the Lagrangian and state Lagrange's equations for this system subject to the force of gravity at the surface of the earth.

(b) Find the motion of the system by integrating Lagrange's equation, given that all objects are initially at rest and the sphere's center is at a distance H above the surface.

(UC, Berkeley)

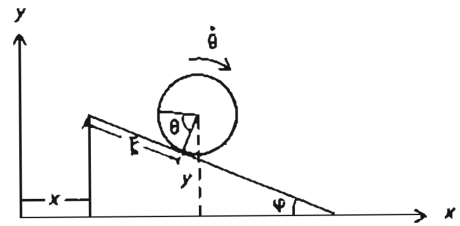


Fig. 2.18.

Solution:

(a) Use a fixed coordinate frame as shown in Fig. 2.18 and let θ be the angle of rotation of the sphere. As the sphere rolls without slipping down the inclined plane, its center will have coordinates

$$(x + (\xi_0 + R\theta)\cos\varphi, H - R\theta\sin\varphi)$$

and velocity

$$(\dot{x} + R\dot{\theta}\cos\varphi, -R\dot{\theta}\sin\varphi).$$

Note that at $t=0$, $x=0$, $\theta=0$, $\xi=\xi_0$, $y=H$, $\dot{x}=\dot{\theta}=0$. Then the Lagrangian is

$$L = T - V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}M(\dot{x}^2 + R^2\dot{\theta}^2 + 2R\dot{x}\dot{\theta}\cos\varphi) + \frac{1}{5}MR^2\dot{\theta}^2 - Mg(H - R\theta\sin\varphi).$$

Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

give

$$(m+M)\ddot{x} + MR\ddot{\theta}\cos\varphi = 0,$$
$$\ddot{x}\cos\varphi + \frac{7}{5}R\ddot{\theta} - g\sin\varphi = 0.$$

(b) Eliminating $\ddot{x}$ from the above gives

$$\left(\frac{7}{5} - \frac{M\cos^2\varphi}{m+M}\right)\ddot{\theta} = \frac{g\sin\varphi}{R},$$

and the above equation becomes

$$\ddot{\rho} + \frac{3m_2 g}{(m_1+m_2)(l-d)}\rho = 0 .$$

Hence ρ oscillates about O, i.e. r oscillates about the value $l-d$, with angular frequency

$$\omega = \sqrt{\frac{3m_2 g}{(m_1+m_2)(l-d)}} ,$$

or period

$$T = 2\pi\sqrt{\frac{(m_1+m_2)(l-d)}{3m_2 g}} .$$

2024

Two rods AB and BC, each of length a and mass m, are frictionlessly joined at B and lie on a frictionless horizontal table. Initially the two rods (i.e. point A, B, C) are collinear. An impulse $\overline{P}$ is applied at point A in a direction perpendicular to the line ABC. Find the motion of the rods immediately after the impulse is applied.

(Columbia)

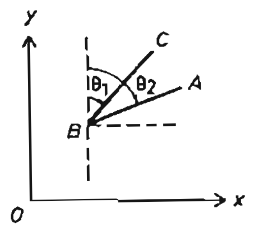


Fig. 2.20.

Solution:

As the two rods AB, BC are freely joined at B, take coordinates as shown in Fig. 2.20 and let the coordinates of B be (x, y). Then the center of mass of BC has coordinates

$$\left(x + \frac{1}{2}a\sin\theta_1, y + \frac{1}{2}a\cos\theta_1\right)$$

and velocity

$$\left(\dot{x} + \frac{1}{2}a\dot{\theta}_1\cos\theta_1, \dot{y} - \frac{1}{2}a\dot{\theta}_1\sin\theta_1\right) ,$$

and that of AB has coordinates

$$\left(x + \frac{1}{2}a\sin\theta_2, y + \frac{1}{2}a\cos\theta_2\right)$$

and velocity

$$\left(\dot{x} + \frac{1}{2}a\dot{\theta}_2\cos\theta_2, \dot{y} - \frac{1}{2}a\dot{\theta}_2\sin\theta_2\right) .$$

Each rod has a moment of inertia about its center of mass of $ma^2/12$. Hence the total kinetic energy is

$$T = \frac{1}{2}m\left[\dot{x}^2 + \dot{y}^2 + \frac{1}{4}a^2\dot{\theta}_1^2 + a\dot{\theta}_1(\dot{x}\cos\theta_1 - \dot{y}\sin\theta_1)\right] + \frac{1}{24}ma^2\dot{\theta}_1^2$$
$$+ \frac{1}{2}m\left[\dot{x}^2 + \dot{y}^2 + \frac{1}{4}a^2\dot{\theta}_1^2 + a\dot{\theta}_2(\dot{x}\cos\theta_2 - \dot{y}\sin\theta_2)\right] + \frac{1}{24}ma^2\dot{\theta}_2^2$$
$$= \frac{1}{2}m\left[2(\dot{x}^2 + \dot{y}^2) + a\dot{x}(\dot{\theta}_1\cos\theta_1 + \dot{\theta}_2\cos\theta_2) - a\dot{y}(\dot{\theta}_1\sin\theta_1 + \dot{\theta}_2\sin\theta_2)\right]$$
$$+ \frac{1}{6}ma^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) .$$

The impulse $\overline{P}$ is applied at A in a direction perpendicular to the line ABC. Thus the virtual moment of the impulse is $\overline{P}\delta(y + a\cos\theta_2)$ and the generalized components of the impulse are

$$Q_x = 0, \qquad Q_y = \overline{P}, \qquad Q_{\theta_1} = 0, \qquad Q_{\theta_2} = -a\overline{P}\sin\theta_2 .$$

Lagrange's equations for impulsive motion are

$$\left(\frac{\partial T}{\partial \dot{q}_j}\right)_f - \left(\frac{\partial T}{\partial \dot{q}_j}\right)_i = Q_j ,$$

where i, f refer to the initial and final states of the system relative to the application of impulse. Note that at $t = 0$ when the impulse is applied, $\theta_1 = -\pi/2$, $\theta_2 = \pi/2$. Furthermore, for the initial state, $\dot{\theta}_1 = \dot{\theta}_2 = \dot{x} = \dot{y} = 0$. As