

Mechanics

Physics 151

Lecture 4
Hamilton's Principle
(Chapter 2)

Administrivia

- Problem Set #1 due
 - Solutions will be posted on the web after this lecture
- Problem Set #2 is here
 - Due next Thursday
- Next lecture (Tuesday) will be given by Srinivas and Abdol-Reza
 - I will be attending a workshop at Stanford

What We Did Last Time


- Derived **Lagrange's Eqn** from Newton's Eqn
 - Using **D'Alembert's Principle** = differential approach
- **Lagrange's Equations** work if
 - Constraints are **holonomic** → Generalized coordinates
 - Forces of constraints do no work → **No frictions**
 - Other forces are **monogenic** → Generalized potential

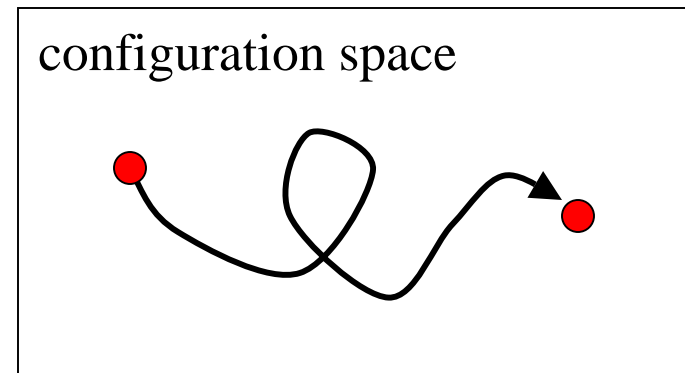
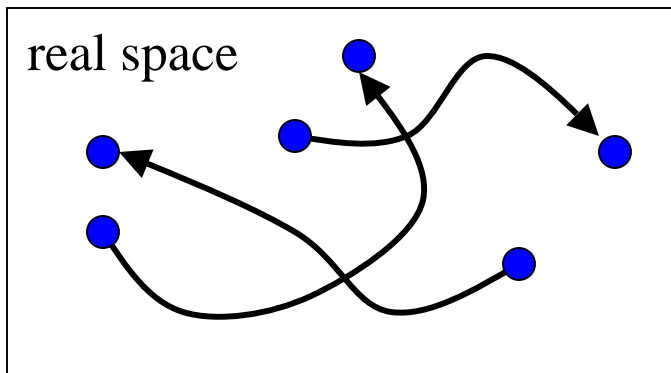
$$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt} \left(\frac{\partial U}{\partial \dot{q}_j} \right)$$

Today's Goals

- Discuss **Hamilton's Principle**
 - Derive Lagrange's Eqn from Hamilton's Principle
 - **Calculus of variation**
 - Looks unfamiliar, but not so difficult
- Discuss **conservation laws** again
 - Using Lagrangian formalism
 - Linear, angular momenta
 - Connection between **symmetry**, **invariance** of the Lagrangian, and **conservation** of generalized momentum

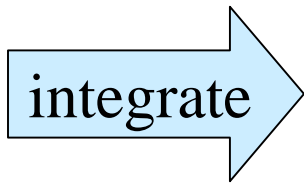
Configuration Space

- Generalized coordinates q_1, \dots, q_n fully describe the system's **configuration** at any moment
- Imagine an n -dimensional space 
 - Each point in this space (q_1, \dots, q_n) corresponds to one configuration of the system
 - Time evolution of the system \rightarrow A curve in the configuration space



Action Integral

- A system is moving as $q_j = q_j(t) \quad j = 1 \dots n$
- Lagrangian is $L(q, \dot{q}, t) = L(q(t), \dot{q}(t), t) = L(t)$



$$I = \int_{t_1}^{t_2} L dt$$

Action, or action integral

- Action I depends on the entire path from t_1 to t_2
- Choice of coordinates q_j does not matter
 - Action is invariant under coordinate transformation

Hamilton's Principle

The action integral of a physical system is *stationary* for the actual path

- This is equivalent to Lagrange's Equations

- We will prove this

We will also define "stationary"

- Three equivalent formulations

- Newton's Eqn depends explicitly on *x-y-z coordinates*

- Lagrange's Eqn is same for any *generalized coordinates*

- Hamilton's Principle refers to *no coordinates*

- Everything is in the action integral

Hamilton's Principle is more fundamental

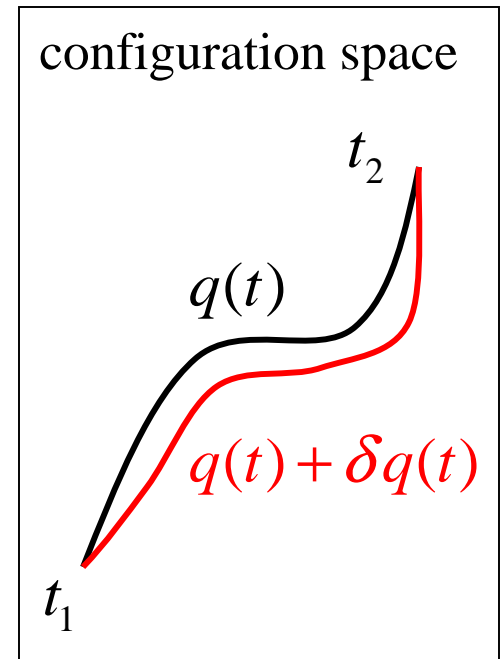
probably...

Stationary

- Consider two paths that are close to each other
 - Difference is infinitesimal
- **Stationary** means that the difference of the action integrals is zero to the 1st order of $\delta q(t)$
 - Similar to “first derivative = 0”

$$\delta I = \int_{t_1}^{t_2} L(q + \delta q, \dot{q} + \delta \dot{q}, t) dt - \int_{t_1}^{t_2} L(q, \dot{q}, t) dt = 0$$

- Almost same as saying “minimum”
 - It could as well be maximum



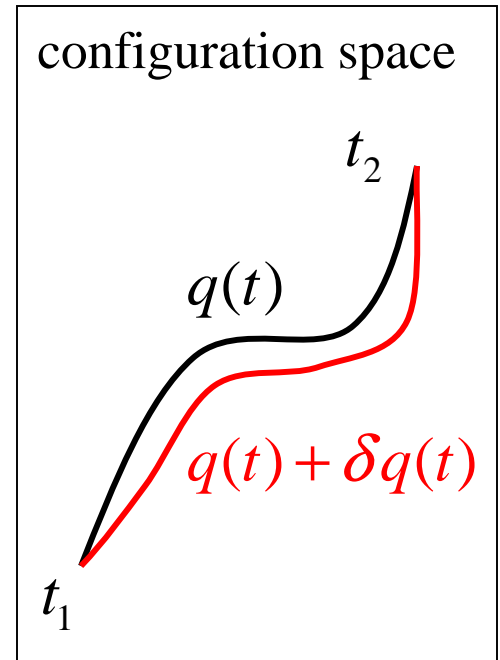
$$\delta q(t_1) = \delta q(t_2) = 0$$

Infinitesimal Path Difference

- What's $\delta q(t)$?
 - It's arbitrary ... sort of
 - It has to be zero at t_1 and t_2
 - It's **well-behaving**

Continuous, non-singular,
continuous 1st and 2nd derivatives

Don't worry
too much



- Have to shrink it to zero
 - Trick: write it as $\delta q(t) = \alpha \eta(t)$
 - α is a parameter, which we'll make $\rightarrow 0$
 - $\eta(t)$ is an arbitrary **well-behaving** function $\eta(t_1) = \eta(t_2) = 0$

Hamilton \rightarrow Lagrange

- Consider 1 generalized coordinate q
 - Add $\delta q(t)$ to $q(t)$, then make $\delta q(t) \rightarrow 0$
 - Do this by $\delta q(t) = \alpha \eta(t)$

$$\rightarrow q(t, \alpha) = q(t) + \alpha \eta(t)$$

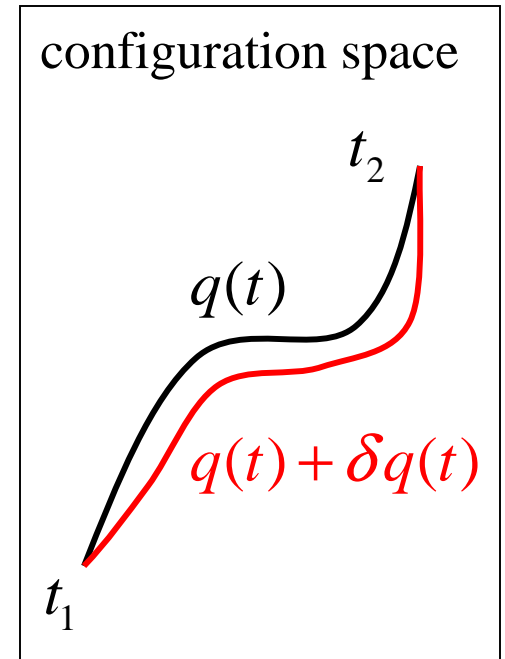
- α is a parameter $\rightarrow 0$
- $\eta(t)$ is an arbitrary **well-behaving** function

$$\eta(t_1) = \eta(t_2) = 0$$

Continuous, non-singular,
continuous η' and η''

- Let's define $I(\alpha) \equiv \int_{t_1}^{t_2} L(q(t, \alpha), \dot{q}(t, \alpha), t) dt$

NB: this also
depends on $\eta(t)$

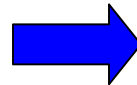


Calculus of Variations

■ Let's define $I(\alpha) = \int_{t_1}^{t_2} L(q(t, \alpha), \dot{q}(t, \alpha), t) dt$

NB: this also depends on $\eta(t)$

■ If the action is stationary



$$\left(\frac{dI(\alpha)}{d\alpha} \right)_{\alpha=0} = 0$$

for any $\eta(t)$

$$\frac{dI(\alpha)}{d\alpha} = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} \frac{dq}{d\alpha} + \frac{\partial L}{\partial \dot{q}} \frac{d\dot{q}}{d\alpha} \right) dt$$

$$q(t, \alpha) = q(t) + \alpha\eta(t)$$

Some work!

$$= \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \frac{dq}{d\alpha} dt$$

$$= \eta(t)$$

Arbitrary function

Lagrange's Equation

- Fundamental lemma

$$\int_{x_1}^{x_2} M(x)\eta(x)dx = 0 \text{ for any } \eta(x) \quad \longrightarrow \quad M(x) = 0 \text{ for } x_1 < x < x_2$$

- We got

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \eta(t) dt = 0 \quad \longrightarrow \quad \frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = 0$$

Done!

Notation of Variation

- For shorthand, we use δ for **infinitesimal variation**
 - I.e. α -derivative at $\alpha = 0$

$$\delta I \equiv \left(\frac{dI}{d\alpha} \right)_{\alpha=0} d\alpha = \frac{d}{d\alpha} \left(\int_{t_1}^{t_2} L(q(t, \alpha), \dot{q}(t, \alpha), t) dt \right) d\alpha$$

$$\delta q \equiv \left(\frac{dq}{d\alpha} \right)_{\alpha=0} d\alpha = \eta(t) d\alpha$$

- Hamilton's Principle can be written as

$$\delta I = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) \delta q dt = 0$$

Going Multi-Coordinates

- Trivial to expand $q \rightarrow (q_1, q_2, \dots, q_n)$

- See Goldstein Section 2.3

$$\delta I = \int_{t_1}^{t_2} \sum_i \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i dt = 0$$

= 0 for each i

- Assumption: $\delta q_1, \delta q_2, \dots$ are arbitrary and independent
 - Not true for x - y - z coordinates if there are constraints
 - True for generalized coordinates if the system is holonomic

Hamilton's Principle

$$\delta I = \delta \int_{t_1}^{t_2} L(q, \dot{q}, t) dt = 0$$

- Action I describes the entire motion of the system
 - It is sufficient to derive the equations of motion
- Action I does not depend on the choice of the coordinates
 - Lagrange formalism is coordinate invariant
- Adding dF/dt to L would add $F(t_2) - F(t_1)$ to I
 - It wouldn't affect $\delta I \leftarrow$ Variations are 0 at t_1 and t_2
 - Arbitrariness of L is obvious

$$L' = L + \frac{dF(q, t)}{dt}$$

Calculus of Variation

- Technique has wider applications

- In general for $J = \int_{x_1}^{x_2} f(y(x), y'(x), x) dx$

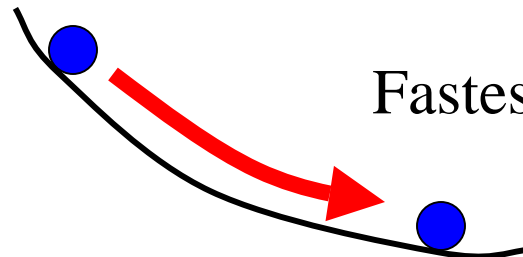
$$y' \equiv \frac{dy}{dx}$$

$$\delta J = 0$$



$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y'} \right) = 0$$

- Examples in Goldstein Section 2.2
- Most famous: the brachistochrone problem



Fastest path via gravity

Conservation Laws

- We've seen (in Lectures 1&2) **conservation of linear, angular momenta and energy** in Newtonian mechanics
 - How do they work with Lagrange's equations?
 - Should better be the same...
- We'll find a few differences and assumptions
 - They are, in fact, limitations we ignored so far

Momentum Conservation

- Let's consider a simple system

$$L = T - V = \sum_i \frac{m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)}{2} - V(x_i, y_i, z_i, t)$$

Potential does not depend on velocity

$$\frac{\partial L}{\partial \dot{x}_i} = m_i \dot{x}_i = p_{ix}$$

Momentum

$$\frac{\partial L}{\partial x_i} = -\frac{\partial V}{\partial x_i} = F_{ix}$$

Force

- Momentum p_{ix} conserved if V does not depend on x_i
- Now try to generalize from here

Generalized Momentum

- Let's call $p_j \equiv \frac{\partial L}{\partial \dot{q}_j}$ the **generalized momentum**
 - Also known as **canonical** or **conjugate momentum**
 - Equals to usual momentum for simple x - y - z coordinates
- Lagrange's equation becomes $\frac{dp_j}{dt} - \frac{\partial L}{\partial q_j} = 0$
 - p_j is conserved if L does not depend explicitly on q_j
 - Such q_j is called **cyclic** (or **ignorable**)

Generalized momentum associated with a cyclic coordinate is conserved

Linear momentum conservation is a special case

Generalized Momentum

$$p_j \equiv \frac{\partial L}{\partial \dot{q}_j}$$

- Generalized momentum may not look like linear momentum
 - Dimension may vary, if q_j is not a space coordinate
 - $p_j q_j$ always has the dimension of action (= work \times time)
 - Form may vary if V depends on velocity
 - Example: a particle in EM field

$$L = \frac{1}{2}mv^2 - q\phi + q\mathbf{A} \cdot \mathbf{v} \quad \longrightarrow \quad p_x = m\dot{x} + qA_x$$

Extra term due to velocity-dependent potential

Symmetry

- Linear momentum $\mathbf{p} = (p_x, p_y, p_z)$ is conjugate of (x, y, z) coordinates
 - Conserved if Lagrangian does not depend explicitly on position
 - I.e. if Lagrangian is invariant under space translation
$$(x, y, z) \rightarrow (x + \Delta x, y + \Delta y, z + \Delta z)$$
 - Such a system is called symmetric under space translation
- Symmetry of a system = Invariance of Lagrangian
→ Conservation of conjugate momentum
 - Let's study an example of angular momentum

Angular Momentum

- Consider a multi-particle system $\mathbf{r}_i = \mathbf{r}_i(q_1, \dots, q_n, t)$
 - Suppose q_1 turns the whole system around
 - Example: ϕ in $\mathbf{r}_i = (x_i, y_i, z_i) = (r_i \cos \phi, r_i \sin \phi, z_i)$
 - Assume V does not depend on $\dot{\phi}$

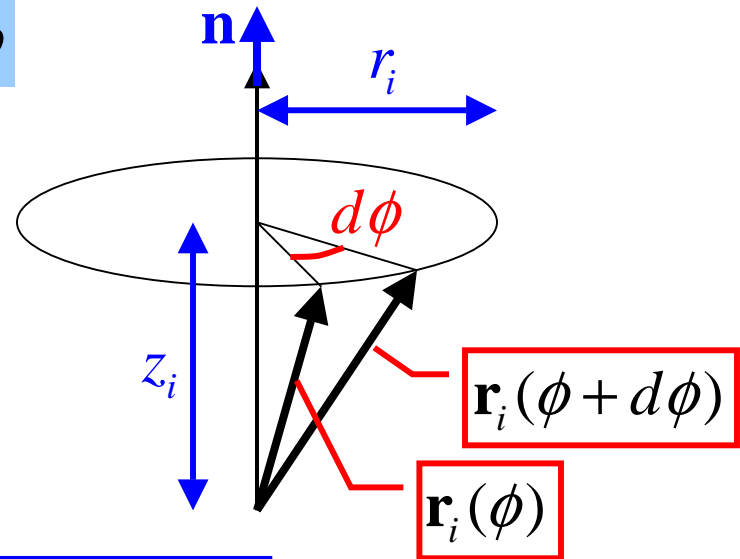
- Conjugate momentum is

$$p_\phi \equiv \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial T}{\partial \dot{\phi}}$$

bit of work \rightarrow $= \mathbf{n} \cdot \sum_i \mathbf{L}_i = \mathbf{n} \cdot \mathbf{L}$

Axis of rotation

Total angular momentum



Bit of Work

$$\mathbf{r}_i = \mathbf{r}_i(\phi, q_2, \dots, q_n, t)$$

$$\dot{\mathbf{r}}_i = \frac{\partial \mathbf{r}_i}{\partial \phi} \dot{\phi} + \sum_{k=2}^n \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t} \quad \rightarrow \quad \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{\phi}} = \frac{\partial \mathbf{r}_i}{\partial \phi}$$

$$T = \sum_i \frac{m_i}{2} \dot{\mathbf{r}}_i \cdot \dot{\mathbf{r}}_i \quad \rightarrow \quad \frac{\partial T}{\partial \dot{\phi}} = \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \dot{\mathbf{r}}_i}{\partial \dot{\phi}} = \sum_i m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial \phi}$$

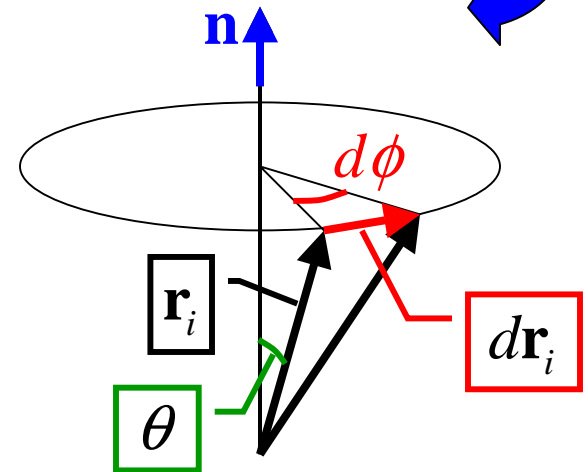
■ $d\mathbf{r}_i$ is perpendicular to both \mathbf{n} and \mathbf{r}_i

■ Size of $d\mathbf{r}_i$ is $r_i \sin \theta d\phi$

$$\frac{\partial \mathbf{r}_i}{\partial \phi} = \mathbf{n} \times \mathbf{r}_i$$

$$= \sum_i m_i \dot{\mathbf{r}}_i \cdot (\mathbf{n} \times \mathbf{r}_i)$$

$$= \sum_i m_i \mathbf{n} \cdot (\mathbf{r}_i \times \dot{\mathbf{r}}_i) = \mathbf{n} \cdot \sum_i \mathbf{L}_i$$



Angular Momentum

- Angular momentum is conserved if the system is symmetric under rotation
 - How does this relate to the total torque \mathbf{N} ?

Generalized force $\rightarrow Q_\phi \equiv \frac{\partial L}{\partial \dot{\phi}}$ \leftarrow This must be zero if ϕ is cyclic

- T cannot depend on $\phi \leftarrow$ Rotating doesn't change v_i^2

$$\frac{\partial L}{\partial \dot{\phi}} = -\frac{\partial V}{\partial \dot{\phi}} = \sum_i \mathbf{F}_i \frac{\partial \mathbf{r}_i}{\partial \dot{\phi}} = \sum_i \mathbf{F}_i \cdot (\mathbf{n} \times \mathbf{r}_i) = \mathbf{n} \cdot \sum_i \mathbf{r}_i \times \mathbf{F}_i$$

Torque

Total torque is zero along the axis of symmetry

Conservation Laws

- Following statements are equivalent:
 - System is symmetric wrt a generalized coordinate
 - The coordinate is cyclic (does not appear in Lagrangian)
 - The conjugate generalized momentum is conserved
 - The associated generalized force is zero

Symmetry	Spatial translation	Rotation
Coordinate	Distance along an axis	Angle around an axis
Momentum	Linear	Angular
Force	Force	Torque

Summary

- Derived Lagrange's Eqn from Hamilton's Principle
 - Calculus of variation
- Discussed conservation laws
 - Generalized (conjugate) momentum $p_j \equiv \frac{\partial L}{\partial \dot{q}_j}$
 - Symmetry of the system
 - Invariance of the Lagrangian
 - Conservation of momentum
- We are almost done with the basic concepts
 - Finish up next Tuesday with energy conservation
 - Some applications are in order → **Central force problem**