16.61 - Aerospace Dynamics Spring 2006
Sample Quiz 2

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Sample Quiz 2
Solution Sketch

Problem 1

As shown in Figure 1, a simple pendulum consisting of a mass m and a massless rod of length l is mounted on a support of mass M which is attached to a horizontal spring with force constant k. The horizontal surface on which the support mass M rests is frictionless, and gravity works in the minus y-direction. Use the Lagrangian approach to find the equations of motion of the system taking (x, θ) as generalized coordinates.
References:
[1] Lim, Problems and Solutions on Mechanics, Part II, Section 2, Problem 2051 (Columbia Physics PhD Exam Question), p 577–579, World Scientific, 1994.

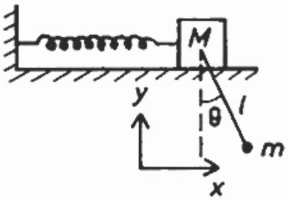

Figure 1: Spring and Mass with Pendulum

Problem 2

As shown in Figure 2, a box of mass m_0 supports a simple pendulum of mass m and length l. A spring of stiffness k forms a horizontal mass-spring system with the box which can slide without friction on a horizontal surface. Use Lagrangian techniques to find the differential equations of motion of the system.
References:
[1] 16.61 OCW, Exam #2, Problem 3, 2003.
[2] Greenwood, Principles of Dynamics, Chapter 6, Problem 6-7, p 276-77, Prentice-Hall, 1965.

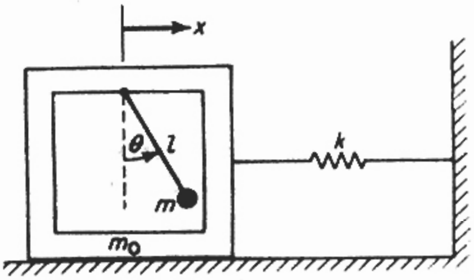

Figure 2: Pendulum in Box

Problem 2 is Problem 1!

Problem 3

Figure 3 shows a homogeneous circular cylinder of radius r, mass m, and mass moment J about its center of mass. The cylinder rolls without slipping on a curved surface of radius R. Taking θ as the generalized coordinate, use the Lagrangian approach to find the equations of motion for this one-degree-of-freedom system.
Hint: Find the velocity v of the center of mass of the cylinder as a function of $(R, r, \dot{\theta})$, and then using the rolling constraint find the angular velocity of the cylinder.
References:
[1] Shabana, Computational Dynamics, Chapter 3, Example 3.12, p 247–248, Wiley, 1994.

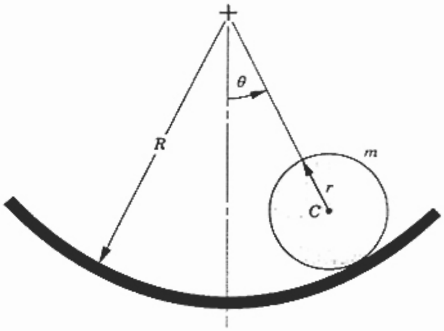

Figure 3: Cylinder Rolling on Curved Surface

Problem 4

As shown in Figure 4, a horizontal turntable rotates at a constant rate ω about a fixed vertical axis through its center O. A particle of mass m can slide in a frictionless circular groove of radius r which is centered at O', which is located a distance $r/3$ from O. Taking θ as defined in the figure as the generalized coordinate, use Lagrangian techniques to find the differential equations of motion.
References:
[1] 16.61 OCW, Exam #2, Problem 1, 2003.

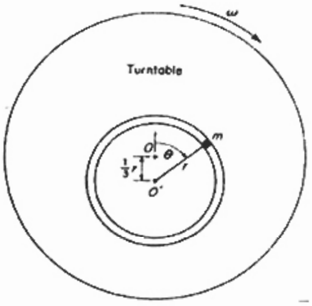

Figure 4: Ball in Track

Problem 4 is solved in a similar manner to Problem 1 in Homework 3. They are essentially the same problem.

$$\frac{1}{12}m\left(\frac{3l}{2}\right)^2=\frac{3}{16}ml^2 .$$

Hence its Lagrangian is

$$\begin{aligned}L&=T-V\\&=\frac{1}{2}ml^2\left[\dot\varphi^2+\frac{9}{16}\dot\theta^2+\frac{3}{2}\dot\theta\dot\varphi\cos(\theta-\varphi)\right]\\&\quad+\frac{3}{32}ml^2\dot\theta^2+mgl\left(\cos\varphi+\frac{3}{4}\cos\theta\right)\\&\approx\frac{1}{2}ml^2\left(\frac{3}{4}\dot\theta^2+\dot\varphi^2+\frac{3}{2}\dot\theta\dot\varphi\right)+\frac{7}{4}mgl-\frac{1}{2}mgl\left(\varphi^2+\frac{3}{4}\theta^2\right)\end{aligned}$$

for small oscillations, retaining only terms of up to the second order of the small quantities $\theta, \varphi, \dot\theta, \dot\varphi$.

Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right)-\frac{\partial L}{\partial q_i}=0$$

give

$$\frac{3}{4}l\ddot\theta+l\ddot\varphi+g\varphi=0 ,$$

$$l\ddot\theta+l\ddot\varphi+g\theta=0 .$$

With a solution of the form

$$\theta=Ae^{i\omega t},\qquad \varphi=Be^{i\omega t} ,$$

the above give

$$\begin{pmatrix}-\frac{3}{4}l\omega^2 & g-l\omega^2\\ g-l\omega^2 & -l\omega^2\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}=0 .$$

The secular equation

$$\begin{vmatrix}-\frac{3}{4}l\omega^2 & g-l\omega^2\\ g-l\omega^2 & -l\omega^2\end{vmatrix}=0 ,$$

i.e.

$$l^2\omega^4-8lg\omega^2+4g^2=0 ,$$

has solutions

$$\omega^2=(4\pm2\sqrt{3})\frac{g}{l}=(1\pm\sqrt{3})^2\frac{g}{l} ,$$

or

$$\omega=(\sqrt{3}\pm1)\sqrt{\frac{g}{l}} ,$$

since ω has to be positive. Hence the normal-mode angular frequencies are

$$\omega_1=(\sqrt{3}+1)\sqrt{\frac{g}{l}},\qquad \omega_2=(\sqrt{3}-1)\sqrt{\frac{g}{l}} .$$

The ratio of amplitudes is

$$\frac{B}{A}=\frac{g-l\omega^2}{l\omega^2}=\begin{cases}-\frac{\sqrt{3}}{2} & \text{for } \omega=\omega_1 ,\\ \frac{\sqrt{3}}{2} & \text{for } \omega=\omega_2 .\end{cases}$$

Thus in the normal-mode given by ω_1, θ and φ are opposite in phase, while in that given by ω_2, θ and φ are in phase. In both cases the ratio of the amplitude of φ to that of θ is

$$\sqrt{3}:2 .$$

2051

A simple pendulum consisting of a mass m and weightless string of length l is mounted on a support of mass M which is attached to a horizontal spring with force constant k as shown in Fig. 2.51.

(a) Set up Lagrange's equations.

(b) Find the frequencies for small oscillations.

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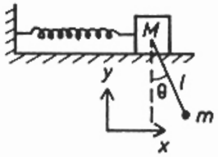

Fig. 2.51.

Solution:

(a) Use coordinates with origin at the position of m when the system is in equilibrium, and the x- and y- axes along the horizontal and vertical directions respectively as shown in Fig. 2.51. Then M and m have coordinates and velocities

$$\begin{aligned} &(x, l), \qquad (x + l\sin\theta, l - l\cos\theta) \\ &(\dot{x}, 0), \qquad (\dot{x} + l\dot{\theta}\cos\theta, l\dot{\theta}\sin\theta) \end{aligned}$$

respectively. The Lagrangian of the system is

$$\begin{aligned} L &= T - V \\ &= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + l^2\dot{\theta}^2 + 2l\dot{x}\dot{\theta}\cos\theta) - Mgl - mgl(1 - \cos\theta) - \frac{1}{2}kx^2 \,. \end{aligned}$$

Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

then give

$$(M + m)\ddot{x} - ml\dot{\theta}^2\sin\theta + ml\ddot{\theta}\cos\theta + kx = 0 \,,$$

$$l\ddot{\theta} + \ddot{x}\cos\theta + g\sin\theta = 0 \,.$$

(b) For small oscillations, $x, \theta, \dot{x}, \dot{\theta}$ are small quantities. Neglecting terms of orders higher than two, the equations of motion become

$$\begin{aligned} (M + m)\ddot{x} + ml\ddot{\theta} + kx &= 0 \,, \\ l\ddot{\theta} + \ddot{x} + g\theta &= 0 \,. \end{aligned}$$

Set

$$x = A\exp(i\omega t) \,, \qquad \theta = B\exp(i\omega t) \,.$$

These equations become

$$\begin{pmatrix} k - (M + m)\omega^2 & -ml\omega^2 \\ -\omega^2 & g - l\omega^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0 \,.$$

The secular equation

$$\begin{vmatrix} k - (M + m)\omega^2 & -ml\omega^2 \\ -\omega^2 & g - l\omega^2 \end{vmatrix} = Ml\omega^2 - [g(M + m) + kl]\omega^2 + gk = 0$$

has two positive roots

$$\omega_1 = \left[\frac{g(M + m) + kl + \sqrt{[g(M + m) + kl]^2 - 4Mlgk}}{2Ml}\right]^{\frac{1}{2}} \,,$$

$$\omega_2 = \left[\frac{g(M + m) + kl - \sqrt{[g(M + m) + kl]^2 - 4Mlgk}}{2Ml}\right]^{\frac{1}{2}} \,,$$

which are the normal-mode angular frequencies of the system.

2052

Two masses, $2m$ and m, are suspended from a fixed frame by elastic springs as shown in Fig. 2.52. The elastic constant (force/unit length) of each spring is k. Consider only vertical motion.

(a) Calculate the frequencies of the normal-modes of oscillations of this system.

(b) The upper mass $2m$ is slowly displaced downwards from the equilibrium position by a distance l and then let go, so that the system performs free oscillations. Calculate the subsequent motion of the lower mass m.

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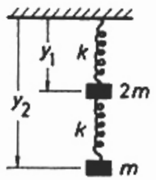

Fig. 2.52.

Solution:

(a) Let the natural lengths of the upper and lower springs be l_1, l_2, and denote the positions of the upper and lower masses by y_1, y_2 as shown in Fig. 2.52, respectively. The Lagrangian is then

Conservation Theorem A generalized coordinate that does not appear in the Lagrangian L is called *cyclic* or *ignorable*. Cyclic or ignorable coordinates are also absent from the Hamiltonian H. For a cyclic coordinate q_k, one has

$$\frac{\partial L}{\partial q_k} = \frac{\partial H}{\partial q_k} = 0$$

If there are no nonconservative forces associated with the cyclic coordinate q_k, Eq. 179 yields

$$\dot{P}_k = 0$$

which implies that the generalized momentum associated with the cyclic coordinate is conserved, that is,

$$P_k = \text{constant}$$

We also note that if all the forces acting on the system are conservative, one has from Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) = \frac{\partial L}{\partial \mathbf{q}} \tag{5.180}$$

The total time derivative of the Lagrangian L is given by

$$\frac{dL}{dt} = \left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)\ddot{\mathbf{q}} + \left(\frac{\partial L}{\partial \mathbf{q}}\right)\dot{\mathbf{q}}$$

Substituting Eq. 180 into the preceding equation, one gets

$$\frac{dL}{dt} = \frac{\partial L}{\partial \dot{\mathbf{q}}}\ddot{\mathbf{q}} + \left\{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right)\right\}\dot{\mathbf{q}}$$

which yields

$$\frac{dL}{dt} = \frac{d}{dt}\left\{\frac{\partial L}{\partial \dot{\mathbf{q}}}\dot{\mathbf{q}}\right\}$$

or equivalently,

$$\frac{d}{dt}\left(L - \frac{\partial L}{\partial \dot{\mathbf{q}}}\dot{\mathbf{q}}\right) = 0 \tag{5.181}$$

Since the potential energy function is independent of the velocities, one has

$$\frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\partial T}{\partial \dot{\mathbf{q}}} = \mathbf{P}^{\mathrm{T}}$$

Using this equation, Eq. 181 can be written as

$$\frac{d}{dt}(L - \mathbf{P}^{\mathrm{T}}\dot{\mathbf{q}}) = \frac{d}{dt}(-H) = 0$$

which implies that

$$H = \mathbf{P}^{\mathrm{T}}\dot{\mathbf{q}} - L = \text{constant}$$

That is, in the case of a conservative system, the Hamiltonian H is a constant of motion. We also note that since

$$\mathbf{P}^{\mathrm{T}}\dot{\mathbf{q}} = \frac{\partial T}{\partial \dot{\mathbf{q}}}\dot{\mathbf{q}} = 2T,$$

the Hamiltonian H takes the following form:

$$H = 2T - L = 2T - T + V = T + V \tag{5.182}$$

which implies that, for a conservative system, the Hamiltonian is the sum of the kinetic and potential energies of the system and it remains constant throughout the system motion.

Example 5.11

Figure 24 shows a homogeneous circular cylinder of radius r, mass m, and mass moment of inertia J about its center of mass, where $J = m(r)^2/2$. The cylinder rolls without slipping on a curved surface of radius R. Use the principle of conservation of energy to derive the equation of motion of the cylinder.

Solution. The kinetic and potential energies of the cylinder are

$$T = \tfrac{1}{2}m(v_c)^2 + \tfrac{1}{2}J(\omega)^2$$
$$V = mg(R - r)(1 - \cos\theta)$$

where v_c is the absolute velocity of the center of mass of the cylinder and ω is its angular velocity, both defined as

$$v_c = (R - r)\dot{\theta}$$

$$\omega = \frac{v_c}{r} = \frac{(R - r)\dot{\theta}}{r}$$

Substituting from these two equatic

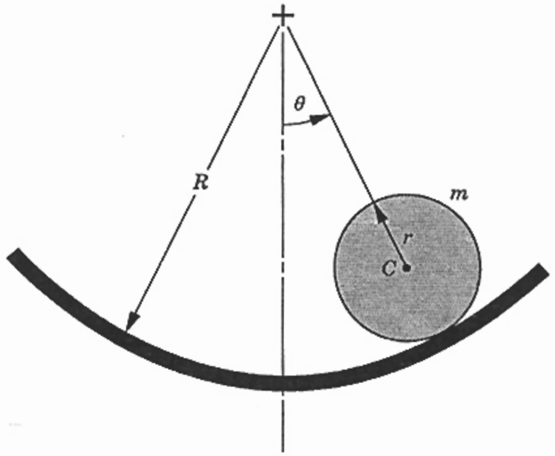

Figure 5.24 Conservation of energy

obtain

$$T = \tfrac{3}{4}\, m(R-r)^2(\dot{\theta})^2$$

The Hamiltonian H is

$$H = T + V = \tfrac{3}{4}\, m(R-r)^2(\dot{\theta})^2 + mg(R-r)(1-\cos\theta)$$

Since the Hamiltonian is constant,

$$\frac{dH}{dt} = \frac{3}{2}\, m(R-r)^2\dot{\theta}\ddot{\theta} + mg(R-r)\dot{\theta}\sin\theta = 0$$

which yields the equation of motion of the cylinder

$$\tfrac{3}{2}(R-r)\ddot{\theta} + g\sin\theta = 0$$

5.11 RELATIONSHIP BETWEEN VIRTUAL WORK AND GAUSSIAN ELIMINATION

The results obtained previoulsy in this chapter for the slider crank mechanism shown in Fig. 20 using the principle of virtual work can also be obtained using the Gaussian elimination and the equations of the static equilibrium. Figure 25 shows the forces acting on the links of the slider crank mechanism. The equations of the static equilibrium of the crankshaft can be written as

$$F_x^{12} - F_x^{23} = 0$$
$$F_y^{12} - F_y^{23} - m^2 g = 0$$
$$F_x^{12} l_O^2 \sin\theta^2 - F_y^{12} l_O^2 \cos\theta^2 + F_x^{23} l_A^2 \sin\theta^2 - F_y^{23} l_A^2 \cos\theta^2 + M^2 = 0$$

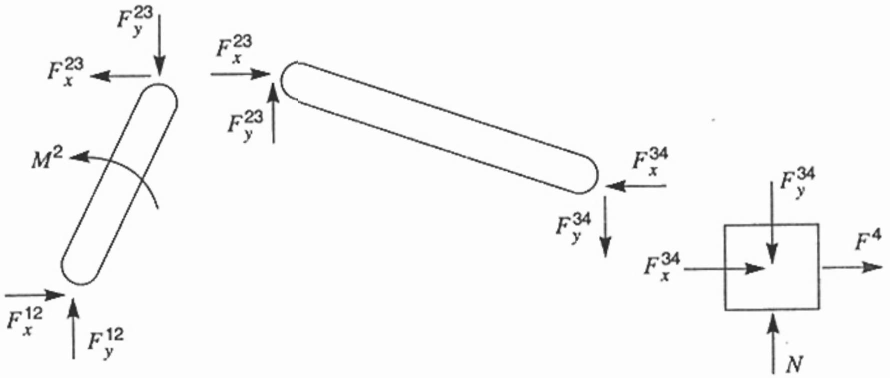

Figure 5.25 Forces of the slider crank mechanism

The equations of the static equilibrium of the connecting rod are

$$F_x^{23} - F_x^{34} = 0$$
$$F_y^{23} - F_y^{34} - m^3 g = 0$$
$$F_x^{23} l_A^3 \sin\theta^3 - F_y^{23} l_A^3 \cos\theta^3 + F_x^{34} l_B^3 \sin\theta^3 - F_y^{34} l_B^3 \cos\theta^3 = 0$$

The equation of the static equilibrium fo the slider block is

$$F_x^{34} + F^4 = 0$$

The preceding seven equations of the static equilibrium of the three links of the slider crank mechanism can be rearranged and written in the following matrix form:

$$\begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ l_O^2\sin\theta^2 & -l_O^2\cos\theta^2 & l_A^2\sin\theta^2 & -l_A^2\cos\theta^2 & 0 & 0 & 1 \\ 0 & 0 & l_A^3\sin\theta^3 & -l_A^3\cos\theta^3 & l_B^3\sin\theta^3 & -l_B^3\cos\theta^3 & 0 \end{bmatrix} \cdot \begin{bmatrix} F_x^{12} \\ F_y^{12} \\ F_x^{23} \\ F_y^{23} \\ F_x^{34} \\ F_y^{34} \\ M^2 \end{bmatrix} = \begin{bmatrix} 0 \\ m^2 \\ 0 \\ m^3 \\ -F \\ 0 \\ 0 \end{bmatrix}$$

In this matrix equation it is assumed that the unknowns are the external moment M^2 and the components of the reaction forces F_x^{12}, F_y^{12}, F_x^{23}, F_y^{23}, F_x^{34}, and F_y^{34}. A standard Gaussian elimination procedure can be used to obtain an upper triangular form of the coefficient matrix in the preceding equation. This Gaussian