Massachusetts Institute of Technology

2.29 Numerical Fluid Mechanics

A Sensitivity Analysis of Stochastic Ocean Forecasting with the Dynamically Orthogonal Primitive Equations

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1 Motivation

The Dynamically Orthogonal Primitive Equations, DO-PE, a reduced-order framework, is able to construct a vast set of initial conditions (in the range of millions) based on a limited but relevant set of recorded temperature and salinity historical vertical profiles (on the order of hundreds.) DO-PE had previously been tested in the Gulf of Mexico and the Alboran Sea, showing excellent results against its full-order counterparts, capturing the rich ocean dynamics in these two locations. The DO-PE framework has mainly three phases: **i)** Ensemble generation, **ii)** Initialization, and **iii)** Execution, i.e., running the DO-PE simulation [1]. The initialization has been chosen to be studied for this project. The DO-PE coefficients are generated based on the Gaussian Mixture Model, GMM. A five-cluster parameter was used in a previous study [1]. While the previous study has provided promising results, a sensitivity analysis was not developed on the optimal number of clusters to generate the coefficient needed for the execution phase. This project aims to develop a sensitivity analysis of stochastic ocean forecasting with the DO-PE framework by studying the optimal cluster parameter.

2 Background

2.1 Stochastic Primitive Equations with free surface

The equations of motion used in our MSEAS (deterministic) modeling system are called the Primitive Equations, PE, which are derived from the mass, momentum (Navier-Stokes), energy, and salt conservation differential equations, with the hydrostatic and Boussinesq approximations [2]. The PE with free surface are given below:

$$\nabla \cdot \mathbf{u}(\mathbf{x}, t; \omega) + \frac{\partial w(\mathbf{x}, t; \omega)}{\partial z} = 0 \tag{1}$$

$$\frac{\partial \mathbf{u}(\mathbf{x}, t; \omega)}{\partial t} + f\hat{k} \times \mathbf{u}(\mathbf{x}, t; \omega) = -\nabla \cdot ([\mathbf{u}(\mathbf{x}, t; \omega) w(\mathbf{x}, t; \omega)]\mathbf{u}(\mathbf{x}, \mathbf{t}; \omega)) \quad - \frac{1}{\rho_0}\nabla p(\mathbf{x}, t; \omega) + \mathcal{F}_{\mathbf{u}}(\mathbf{x}, t; \omega) \tag{2}$$

$$\frac{\partial p(\mathbf{x}, t; \omega)}{\partial z} = -\rho(\mathbf{x}, t; \omega) g \tag{3}$$

$$\frac{\partial T(\mathbf{x}, t; \omega)}{\partial t} = -\nabla \cdot ([\mathbf{u}(\mathbf{x}, \mathbf{t}; \omega) w(\mathbf{x}, t; \omega)] T(\mathbf{x}, t; \omega)) + \mathcal{F}_T(\mathbf{x}, t; \omega) \tag{4}$$

$$\frac{\partial S(\mathbf{x}, t; \omega)}{\partial t} = -\nabla \cdot ([\mathbf{u}(\mathbf{x}, t; \omega) w(\mathbf{x}, t; \omega)] S(\mathbf{x}, t; \omega)) \quad + \mathcal{F}_S(\mathbf{x}, t; \omega) \tag{5}$$

$$\rho(\mathbf{x}, t; \omega) = \rho(z, T(\mathbf{x}, t; \omega), S(\mathbf{x}, t; \omega)) \tag{6}$$

$$\frac{\partial \eta(x, y, t; \omega)}{\partial t} + \nabla \cdot \left(\int_{-H}^{\eta(x,y,t;\omega)} \mathbf{u}(\mathbf{x}, t; \omega) \mathrm{d}z \right) = 0 \tag{7}$$

Where:

- **u** are the velocities components
- T is the temperature
- S is salinity

- η is the sea surface elevation
- p is the pressure
- f is the Coriolis parameter
- ρ is the density
- ρ_0 is the (constant) density from a reference state
- g is the acceleration due to gravity

2.2 Dynamically Orthogonal, DO, and the orthonormality condition

The DO decomposition for a state vector ψ as well as the orthonormality criterion that must be met while the system evolves in time inside its stochastic subdomain is shown below:

$$\psi(\mathbf{x},t;\omega) = \bar{\psi}(\mathbf{x},t) + \sum_{i=1}^{n_{s,\psi}} \phi_i(t;\omega)\widetilde{\psi}_i(\mathbf{x},t) \tag{8}$$

$$\left\langle \widetilde{\psi}_i(\mathbf{x},t;\omega), \widetilde{\psi}_j(\mathbf{x},t;\omega) \right\rangle = \delta_{ij} \forall (i,j) \tag{9}$$

2.3 DO Mean Equations

The DO framework is applied to the PE equations, and the DO Mean, DO Mode, and DO Coefficients equations are generated. For simplicity, the DO Mode equations are presented in this section [3].

$$\nabla \cdot \bar{\mathbf{u}} + \frac{\partial \bar{w}}{\partial z} = 0 \tag{10}$$

$$\begin{aligned} \frac{\partial \bar{\mathbf{u}}}{\partial t} - \frac{\partial}{\partial z}\left(\kappa \frac{\partial \bar{\mathbf{u}}}{\partial z}\right) + f\hat{k} \times \bar{\mathbf{u}} = -\nabla \cdot (\bar{\mathbf{u}}\bar{\mathbf{u}}) - \frac{\partial (\bar{w}\bar{\mathbf{u}})}{\partial z} + C_{\phi_m \phi_n}\left(-\nabla \cdot (\widetilde{\mathbf{u}}_m \widetilde{\mathbf{u}}_n) - \frac{\partial (\widetilde{w}_m \widetilde{\mathbf{u}}_n)}{\partial z}\right) \\ - \frac{1}{\rho_0}\nabla \overline{p_h} - g\nabla\bar{\eta} + \overline{\mathcal{F}}^u \end{aligned} \tag{11}$$

$$\frac{\partial \overline{p_h}}{\partial z} = -\bar{\rho} g \tag{12}$$

$$\overline{\mathrm{U}} = \frac{1}{H + \bar{\eta}} \int_{-H}^{\bar{\eta}} \bar{\mathbf{u}} \mathrm{d}z \tag{13}$$

$$\frac{\partial \bar{\eta}}{\partial t} + \nabla \cdot ((H + \bar{\eta})\overline{\mathbf{U}}) = 0 \tag{14}$$

$$\frac{\partial \bar{T}}{\partial t} - \frac{\partial}{\partial z}\left(\kappa^{TS} \frac{\partial \bar{T}}{\partial z}\right) = -\nabla \cdot (\bar{\mathbf{u}}\bar{T}) - \frac{\partial (\bar{w}\bar{T})}{\partial z} + C_{\phi_m \phi_n}\left(-\nabla \cdot \left(\widetilde{\mathbf{u}}_m \widetilde{T}_n\right) - \frac{\partial \left(\widetilde{w}_m \widetilde{T}_n\right)}{\partial z}\right) + \overline{\mathcal{F}}^T \tag{15}$$

$$\frac{\partial \bar{S}}{\partial t} - \frac{\partial}{\partial z}\left(\kappa^{TS} \frac{\partial \bar{S}}{\partial z}\right) = -\nabla \cdot (\bar{\mathbf{u}}\bar{S}) - \frac{\partial (\bar{w}\bar{S})}{\partial z} + C_{\phi_m \phi_n}\left(-\nabla \cdot \left(\widetilde{\mathbf{u}}_m \widetilde{S}_n\right) - \frac{\partial \left(\widetilde{w}_m \widetilde{S}_n\right)}{\partial z}\right) + \overline{\mathcal{F}}^S \tag{16}$$

$$\bar{\rho} = \rho(z, \bar{T}, \bar{S}) \tag{17}$$

2.4 DO-PE Framework in the Gulf of Mexico

The Gulf of Mexico Loop Current transports large water masses through the Yucatan Channel into the Gulf of Mexico. This loop impacts the weather and climate of the region and impacts the health of local ecosystems, as it is the dominant process in the Gulf of Mexico [4].

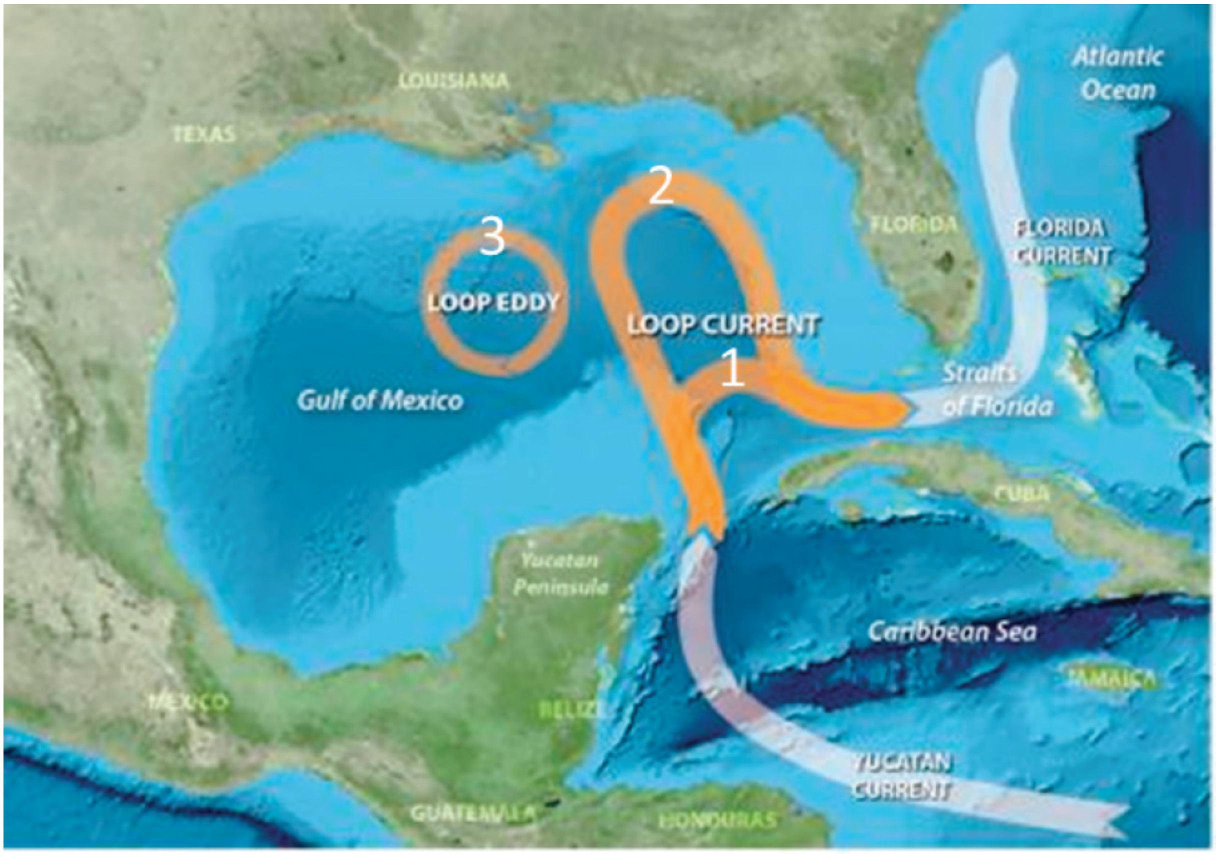

Figure 1: The three possible Loop Current States: (1) Retracted State, (2) Extended State, (3) Loop Current has shed a warm-core anticyclonic Loop Current Eddy (LCE) [4]

2.5 DO-PE Initialization Process

The statistics are based on a significantly smaller Monte Carlo ensemble of (full-order) PE forecasts in order to initialize a (reduced-order) DO-PE forecast ensemble. Based on the analysis of synoptic historical data in conjunction with appropriate correlation functions, each member of that (full order) PE ensemble is obtained by perturbing a central forecast [5]. Dominant modes are derived from the PE ensemble and DO coefficients from the statistics of the PE Ensemble fitted with a Gaussian Mixture Model (GMM).

2.6 Gaussian Mixture Models

Generally, a Gaussian Mixture is a function that is comprised of several Gaussians, **K** [6]. The following parameters characterize each Gaussian in the mixture:

$$\theta = \{\pi, \mu, \Sigma\} \tag{18}$$

Where:

- μ is the mean

- π is the mixing probability
- Σ is the covariance

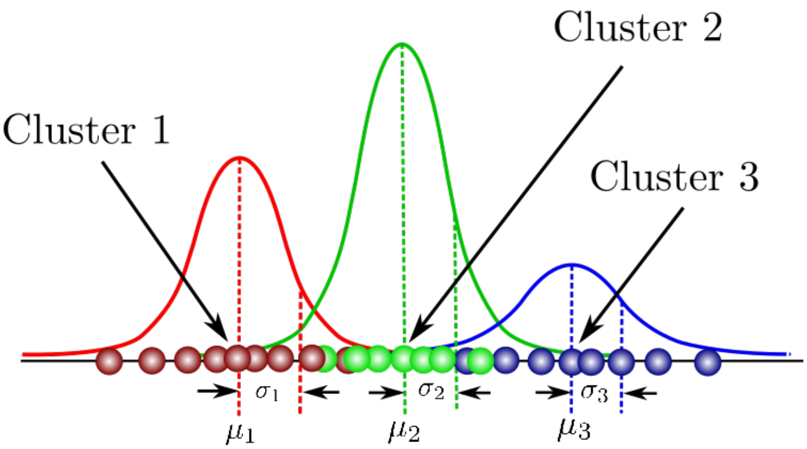

Figure 2: Gaussian functions, hence K = 3 [6]

3 Discussion and Results

The generation of new DO-PE initialization conditions by varying the number of clusters in the GMM framework was partially achieved for this project. The methodology to generate the DO-PE ICs was followed. However, due to time constraints and delays in understanding the functionality of the MSEAS cluster, only the methodology was well understood. While this project has not achieved its primary goal, it is essential to mention that by understanding the methodology, the completion of this project becomes palpable in the foreseeable future. Nevertheless, this project aims to replicate the GMM performances as shown below [1].

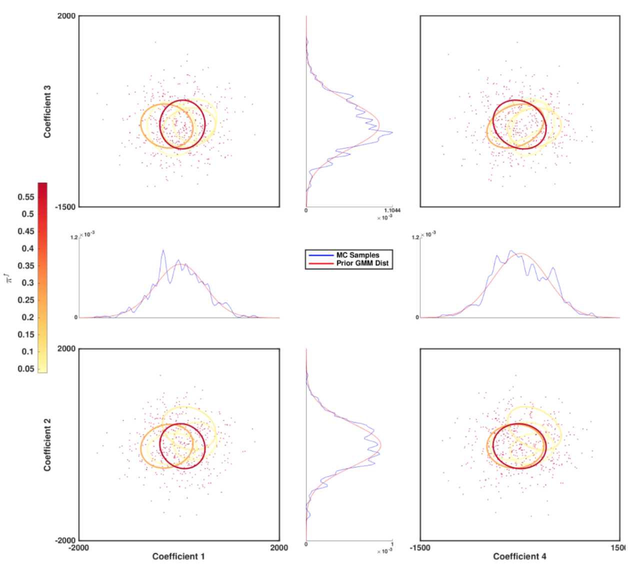

Figure 3: GMM Performance (Modes 1-4) [1]

4 Conclusion

The goal of this 2.29 project was to develop a sensitivity analysis of stochastic ocean forecasting with dynamically orthogonal primitive equations. While the main objective was not met, I have learned some valuable information about DO-PE:

- A comparison of the DO-PE framework and its deterministic counterpart reveals good accuracy.
- DO-PE framework is versatile.
- DO-PE framework could potentially have an immediate impact.
- GMM parameters have shown an avenue of investigation for my research at MSEAS.

References

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