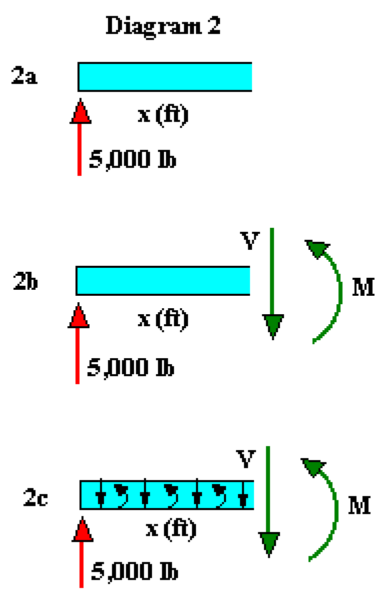
Pset #5 Solutions

3.11 Fall 2003

Solution Problem #1

To understand the shear forces and bending moments in a beam, we will look at a simple problem. We are looking at a simply supported 20 ft. beam with a load of 10,000 lb. acting downward right at the center of the beam. Due to symmetry the two support forces will be equal, with a value of 5000 lb. each. This is the static equilibrium condition for the whole beam.

Next let's examine a section of the beam. We will cut the beam a arbitrary distance (x) between 0 and 10 feet, and apply static equilibrium conditions to the left end section as shown in Diagram 2 below. We can do this since as the entire beam is in static equilibrium, then a section of the beam must also be in equilibrium.



In Diagram 2a, we have shown left section of the beam, x feet, long - where x is an arbitrary distance greater than 0 ft. and less than 10 ft. Notice if we just include the 5000 lb. external support force, the section of the beam is clearly not in equilibrium. Neither the sum of forces (translational equilibrium), nor the sum of torque (rotational equilibrium) will sum to zero - as required for equilibrium. Therefore, since we know the **beam section is in equilibrium**, there must be some forces and/or torque not accounted for.
In diagram 2b, we have shown the missing force and torque. The 10,000 lb.

load which we originally applied to the beam, and the support force cause internal "shearing forces" and internal torque called "bending moments" to develop. (We have symbolically shown these in Diagram 2c.) When we cut the beam, the internal shear force and bending moment at that point then become an external force and moment (torque) acting on the section. We have shown these in Diagram 2b, and labeled them **V (shear force)** and **M (bending moment).**

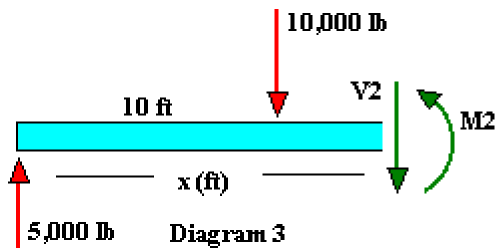
Please note that M is a moment or torque - not a force. It does not appear in the sum of forces equation when we apply static equilibrium to the section - which will be our next step.

Equilibrium Conditions:
Sum of Forces in y-direction: **+ 5000 lb. - V = 0** , solving **V = 5000 lb.**
Sum of Toque about left end: **-V * x + M = 0** , we next substitute the value of V from the force equation into the torque equation: **- 5000 lb. * x + M = 0** , then solving for **M = 5000x (ft-lb.)**

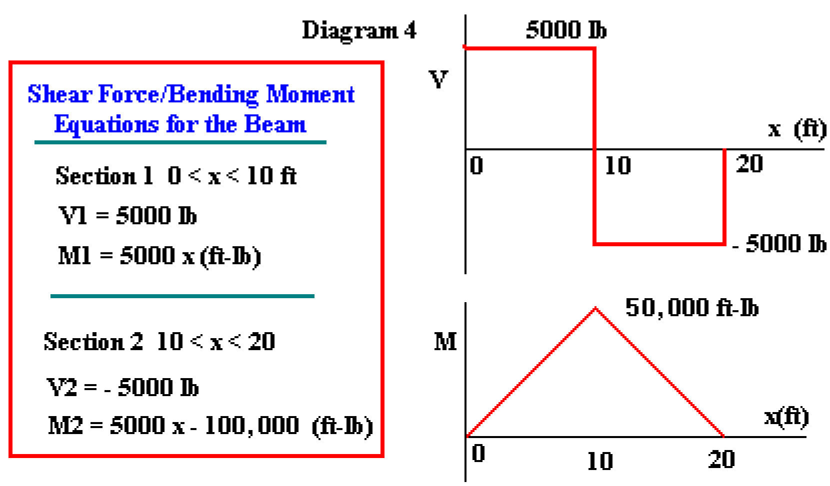
These are the equations for the shear force and bending moments for the section of the beam from 0 to 10 feet. Notice that the internal shear force is a constant value of 5000 lb. for the section, but that the value of the internal torque (bending moment) varies from 0 ft-lb. at x = 0, to a value of 50,000 ft-lb. at x = 10 ft.

[We really should not put exactly 0 ft., and 10 ft. into our equation for the bending moment. The reason is that at 0 and 10 ft., there are 'point loads/forces' acting. That is, we have our forces acting at point - and a point has zero area, so the stress (F/A) at these points would in theory be infinite. Of course, a stress can not be infinite, and we can not apply a force at a point - it is actually applied over some area (even if the area if small). However, in 'book' problems we normally apply forces at a point. To deal with this difficulty, we actually skip around these points. We cut our section at 0' < x < 6'. Still when we put values into our expressions we put in values such as x = 9.99999999 ft, and round it off (numerically) to 10 ft. This is, in effect, cheating a bit. We are putting in the value x = 10 ft., but only because the number we actually put in was rounded off to 10 ft. It all may sound confusing, but it works, and will become clear as we do several examples.]

First, however we will finish analyzing our simple beam. So far we have found expressions for the shear force and bending moments **(V1 = 5000 lb, M1 = 5000x ft-lb)** for section 1 of the beam, between 0 and 10 ft. Now we will look at the next section of the beam. We cut the beam at distance x (ft) from the left end, where x is now greater than 10 ft. and less then 20 ft. and then look at entire section to the left of where we cut the beam (See Diagram 3). Where the beam was cut, we have an internal shear force and bending moment - which now become external. These are shown in Diagram 3 as V2 and M2. (We add the '2', to indicate we are looking at section two of the beam.)



We next apply static equilibrium conditions to the beam section, and obtain:
Equilibrium Conditions:
Sum of Forces in y-direction: + 5000 lb. -10,000 lb. - V = 0 , solving **V2 = - 5000 lb.**
Sum of Toque about left end: -10,000 lb * 10 (ft) -V * x (ft) + M = 0 , we next substitute the value of V from the force equation into the torque equation : - 10,000 lb * 10 ft. - (-5000 lb) * x (ft) + M = 0 , then solving for **M2 = -[5000x (ft-lb.) - 100,000] ft-lb.**
The two expressions above give the value of the internal shear force and bending moment in the beam, between the distances of the 10 ft. and 20 ft. A useful way to visualize this information is to make **Shear Force and Bending Moment Diagrams** - which are really the graphs of the shear force and bending moment expressions over the length of the beam. (See Diagram 4.)



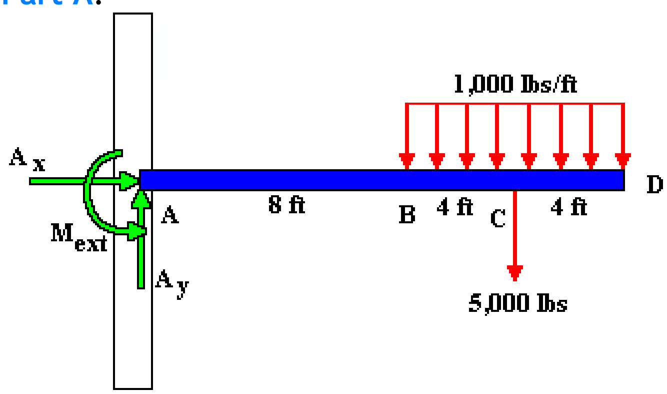
These are a quite useful way of visualizing how the shear force and bending moments vary through out the beam. We have completed our first Shear Force/Bending Moment Problem. We have determined the expressions for the shear forces and bending moments in the beam, and have made accompanying shear force and bending moment diagrams.

MORE DIFFICULT PROBLEMS #2-4

Solution Problem #2

Solution:

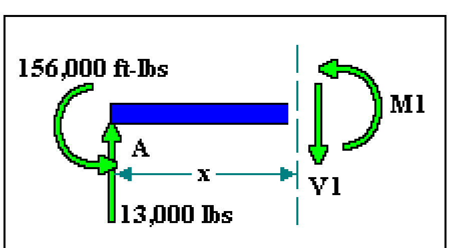
Part A:



STEP 1: Draw a free body diagram showing and labeling all load forces and support (reaction) forces, as well as any needed angles and dimensions.
STEP 2: Break any forces not already in x and y direction into their x and y components.
STEP 3: Apply the equilibrium conditions.

Sum $F_x = A_x = 0$
Sum $F_y = -5{,}000$ lbs $- (1{,}000$ lbs/ft$)(8$ ft$) + A_y = 0$
Sum $T_A = -(5{,}000$ lbs$)(12$ ft$) - (8{,}000$ lbs$)(12$ ft$) + M_{ext} = 0$
Solving for the unknowns:
$A_y = 13{,}000$ lbs; $M_{ext} = 156{,}000$ ft-lbs

Part B: Determine the Shear Forces and Bending Moments expressions for each section of the loaded beam. For this process we will 'cut' the beam into sections, and then use Statics - Sum of Forces to determine the Shear Force expressions, and Integration to determine the Bending Moment expressions in each section of the beam.



Section 1:
Cut beam between 0 and 8 ft.

Section 1: Cut the beam at x, where 0 < x < 8 ft. Analyze left hand section.

1. FBD. (Shown in Diagram)
2. All forces in x & y components (yes)
3. Apply translational equilibrium conditions (forces only):

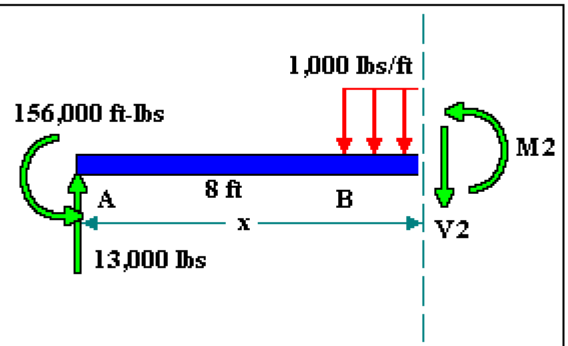
Sum Fx = 0 (no net external x- forces)
Sum Fy = 13,000 lbs - V1 = 0
Solving: V1 = 13,000 lbs

4. Integration $M1 = \int V1\,dx = \int 13{,}000dx$

M1 = 13,000x + C1

a)Boundary condition to find C1: at x=0 M=-156,000 ft-lbs (That is, for a cantilever beam, the value of the bending moment at the wall is equal to the negative of the external moment.)
Apply BC: -156,000 = 13000(0) + C1

Solving: C1 = -156,000
Therefore… M1 = [13,000x - 156,000] ft-lbs for 0 < x < 8 ft.



Section 2: Cut beam between 8 and 12 ft.

Section 2: Cut the beam at x, where 8 < x < 12 ft. Analyze left hand section.

1. FBD. (Shown in Diagram)
2. All forces in x & y components (yes)
3. Apply translational equilibrium conditions (forces only):

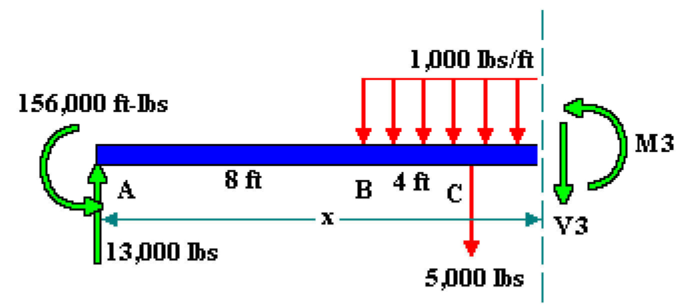
Sum Fx = 0 (no net external x- forces)
Sum Fy = 13,000 lbs - (1,000 lbs/ft)((x - 8)ft) - V2 = 0
Solving: V2 = [-1,000x + 21,000] lbs

4. Integration $M2 = \int V2\,dx = \int [-1{,}000x + 21{,}000]\,dx$

$M2 = -500x^2 + 21{,}000x + C2$

a)Boundary condition to find C2: at x=8 ft M=-52,000 ft-lbs (from equation M1)
Apply BC: $-52{,}000$ ft-lbs $= -500(8)^2 + 21{,}000(8) + C2$

Solving: C2 = -188,000 ft-lbs
Therefore… $M2 = [-500x^2 + 21{,}000x - 188{,}000]$ ft-lbs for $8 < x < 12$



Section 3: Cut beam between 12 and 16 ft.

Section 3: Cut the beam at x, where $12 < x < 16$ ft. Analyze left hand section.

1. FBD. (Shown in Diagram)
2. All forces in x & y components (yes)
3. Apply translational equilibrium conditions (forces only):

Sum Fx = 0 (no net external x- forces)
Sum Fy = 13,000 lbs - (1,000 lbs/ft)((x - 8)ft) - 5,000 lbs - V3 = 0

Solving: V3 = [-1,000x + 16,000] lbs

4. Integration $M3 = \int V3\,dx = \int [-1{,}000x + 16{,}000]\,dx$

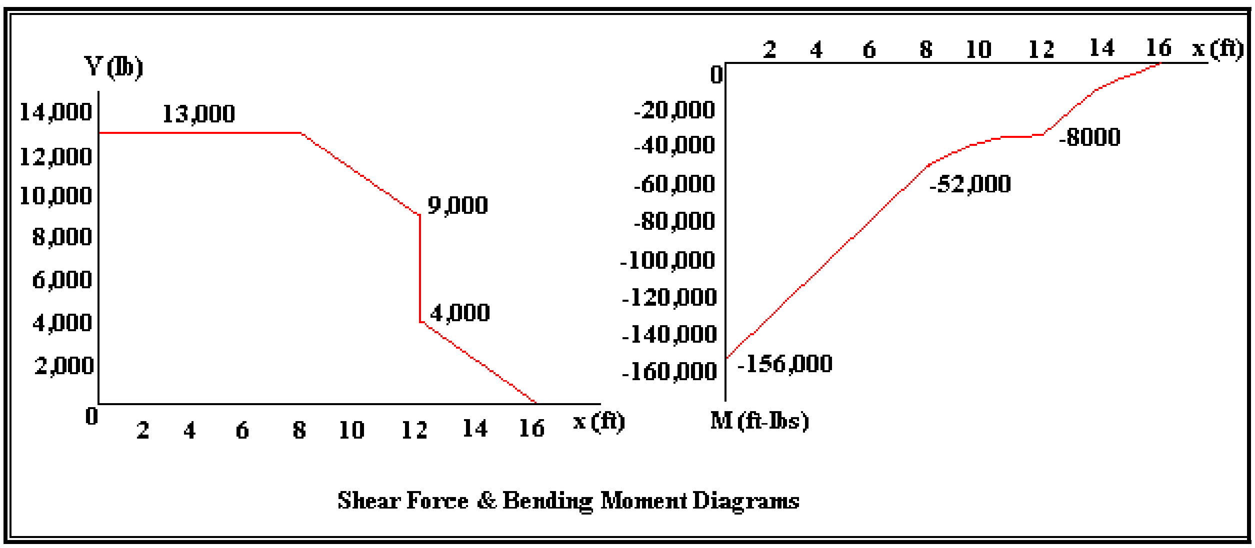
$M3 = -500x^2 + 16{,}000x + C3$

a)Boundary condition to find C3: at x=16 ft M=0 ft-lbs (free end of beam, no external torque so M3=0)
Apply BC: $0 = -500(16)^2 + 16{,}000(16) + C3$

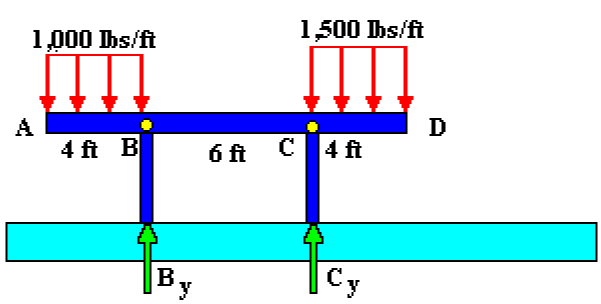
Solving: $C3 = -128,000$ ft-lbs
Therefore… $M3 = [-500x^2 + 16,000x - 128,000]$ ft-lbs for $12 < x < 16$

Part C: Shear Force and Bending Moment Diagrams: Now using the expressions found in Part B above, we can draw the shear force and bending moment diagrams for our loaded beam.

$V1 = 13,000$ lb, $V2 = [-1,000x+21,000]$ lb, $V3 = [-1,000x + 16,000]$ lb
$M1 = [13,000x-156,000]$ ft-lb, $M2 = [-500x^2+21,000x-188,000]$ ft-lb, $M3 = [-500x^2 + 16,000x - 128,000]$ ft-lb



Solution Problem #3

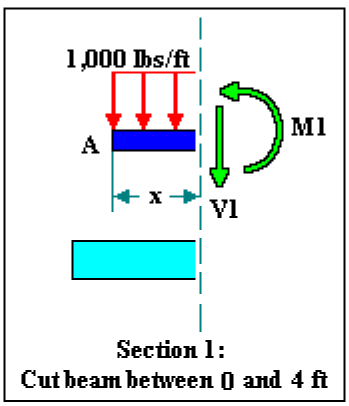


Part A:
STEP 1: Draw a free body diagram showing and labeling all load forces and support (reaction) forces, as well as any

needed angles and dimensions.
STEP 2: Break any forces not already in x and y direction into their x and y components.
STEP 3: Apply the equilibrium conditions.

Sum F_y = (-1,000 lbs/ft)(4 ft) - (1,500 lbs/ft)(4 ft) + B_y + C_y = 0
Sum T_B = (C_y)(6 ft) + (1,000 lbs/ft)(4 ft)(2 ft) - (1,500 lbs/ft)(4 ft)(8 ft) = 0
Solving for the unknowns:
C_y = 6,670 lbs; B_y = 3,330 lbs

Part B: Determine the Shear Forces and Bending Moments expressions for each section of the loaded beam. For this process we will 'cut' the beam into sections, and then use Statics - Sum of Forces to determine the Shear Force expressions, and Integration to determine the Bending Moment expressions in each section of the beam.



Section 1: Cut the beam at x, where 0 < x < 4 ft. Analyze left hand section.

1. FBD. (Shown in Diagram)
2. All forces in x & y components (yes)
3. Apply translational equilibrium conditions (forces only):

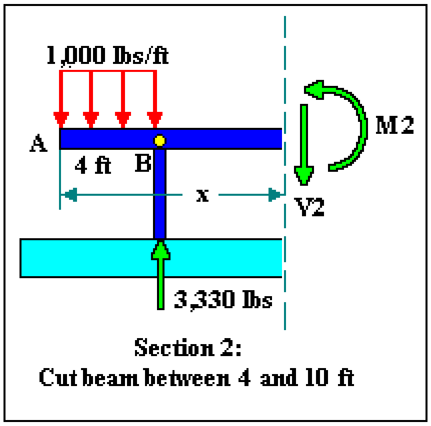
Sum Fx = 0 (no net external x- forces)
Sum Fy = -1,000 lbs/ft(x) - V1 = 0

Solving: V1 = -1,000x lbs

4. Integration $M1 = \int V1\,dx = \int [-1{,}000x]\,dx$

M1 = -500 x^2 + C1

a)Boundary condition to find C1: at x=0 M=0
Apply BC: $0 = -500(0)^2 + C1$
Solving: C1 = 0
Therefore… M1 = [$-500x^2$] ft-lbs for 0 < x < 4 ft.



Section 2:
Cut beam between 4 and 10 ft

Section 2: Cut the beam at x, where 4 < x < 10 ft. Analyze left hand section.

1. FBD. (Shown in Diagram)
2. All forces in x & y components (yes)
3. Apply translational equilibrium conditions (forces only):

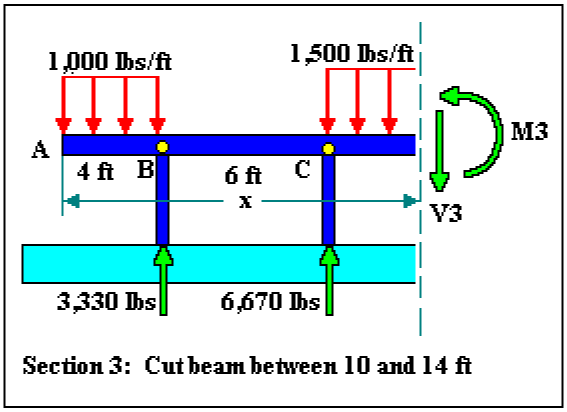
Sum Fx = 0 (no net external x- forces)
Sum Fy = -1,000 lbs/ft (4 ft) + 3,330 lbs - V2 = 0
Solving: V2 = -667 lbs

4. Integration $M2 = \int V2\, dx = \int -667 dx$

M2 = -667x + C2

a)Boundary condition to find C2: at x=4 ft M=-8000 ft-lbs (from equation M1)
Apply BC: 8000 ft-lbs = -667(4) + C2

Solving: C2 = -5,330 ft-lbs
Therefore… M2 = [-667x - 5,330] ft-lbs for 4 < x < 10



Section 3: Cut the beam at x, where 10 < x < 14 ft. Analyze left hand section.

1. FBD. (Shown in Diagram)
2. All forces in x & y components (yes)
3. Apply translational equilibrium conditions (forces only):

Sum Fx = 0 (no net external x- forces)
Sum Fy = -1,000 lbs/ft(4 ft) + 3,330 lbs + 6,670 lbs - 1500lbs/ft(x-10)ft - V3 = 0

Solving: V3 = [-1,500x + 21,000] lbs

4. Integration $M3 = \int V3\,dx = \int [-1{,}500x + 21{,}000]\,dx$

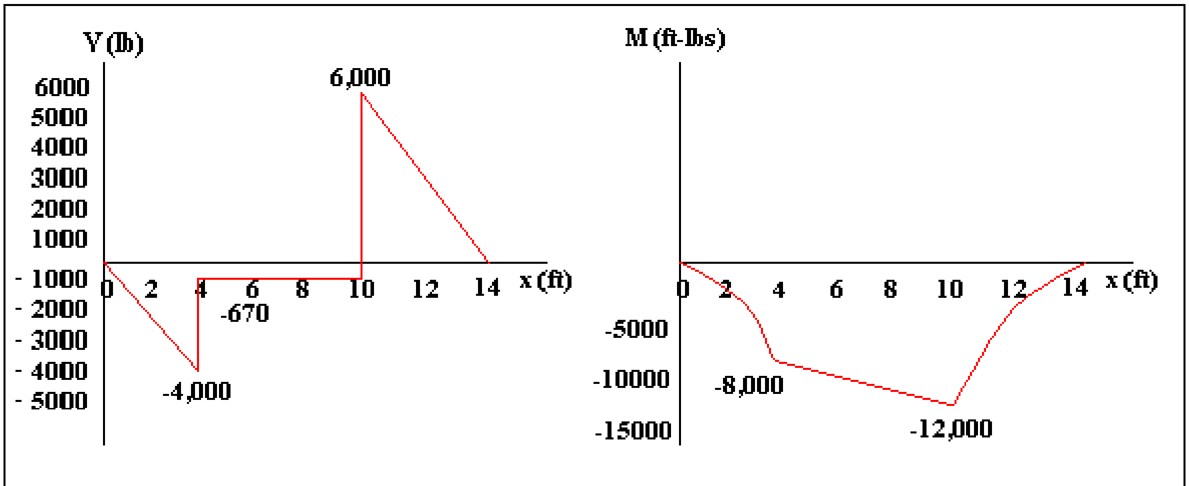
$M3 = -750x^2 + 21{,}000x + C3$

a)Boundary condition to find C3: at x=14 ft M=0 ft-lbs (end of beam, no external torque so M3=0)
Apply BC: $0 = -750(14)^2 + 21{,}000(14) + C3$

Solving: C3 = -147,000 ft-lbs
Therefore… $M3 = [-750x^2 + 21{,}000x - 147{,}000]$ ft-lbs for 10 < x < 14

Part C: Shear Force and Bending Moment Diagrams: Now using the expressions found in Part B above, we can draw the shear force and bending moment diagrams for our loaded beam.

V1 = -1,000x lb, V2 = -667 lb, V3 = -1,500x+21,000 lb
M1 = $-500x^2$ ft-lb, M2 = -667x-5,330 ft-lb, M3 = $-750x^2+21{,}000x-147{,}000$ ft-lb



Shear Force & Bending Moment Diagrams

Solution Problem #4

Overall Equilibrium

We start by drawing a free-body diagram (Fig. 2) of the beam and determining the support reactions. Summing moments about the left end of the beam

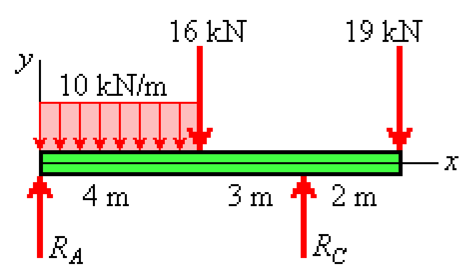


Fig. 2

$$\circlearrowleft \Sigma M_A = 7R_C - 2\,[\,4 \times 10\,] - 4(16) - 9(19) = 0 \qquad \text{(1a)}$$

gives

$$R_C = 45 \text{ kN} \qquad \text{(1b)}$$

Then, summing forces in the vertical direction

$$\uparrow \Sigma F = R_A + R_C - 4 \times 10 - 16 - 19 = 0 \qquad \text{(2a)}$$

gives

$$R_A = 30 \text{ kN} \qquad \text{(2b)}$$

Drawing the Shear Force Diagram

Sometimes we are not so much interested in the equations for the shear force and bending moment as we are in knowing the maximum and minimum values or the values at some particular point. In these cases, we want a quick and efficient method of generating the shear force and bending moment diagrams (graphs) so we can easily find the maximum and minimum values. That is the subject of this first part of the problem.

Concentrated Force

The 30-kN concentrated force (support reaction) at the left end of the beam causes the shear force graph to jump up (in the direction of the force) by 30 kN (the magnitude of the force) from 0 kN to 30 kN.

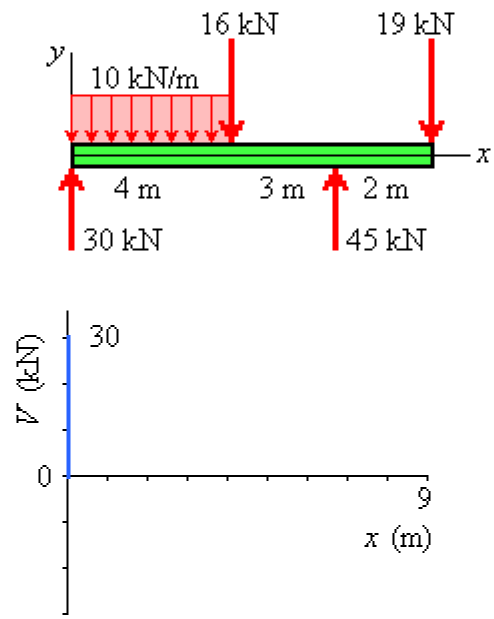


Fig. 3

Distributed Load

The downward distributed load causes the shear force graph to slope downward (in the direction of the load). Since the distributed load is constant, the slope of the shear force graph is constant ($dV/dx = w =$ constant).

The total change in the shear force graph between points *A* and *B* is 40 kN (equal to the area under the distributed load between points *A* and *B*) from +30 kN to -10 kN.

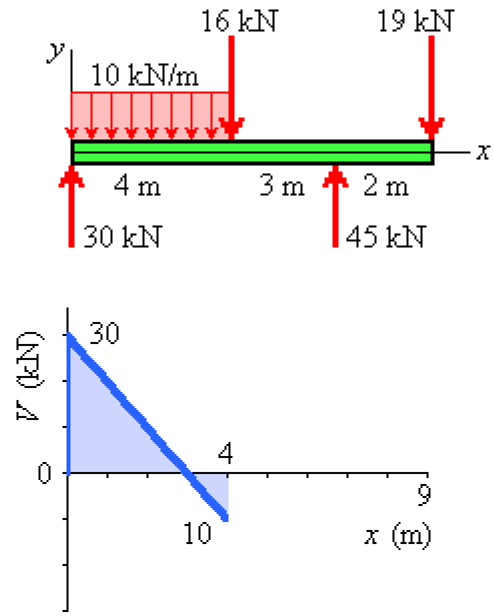


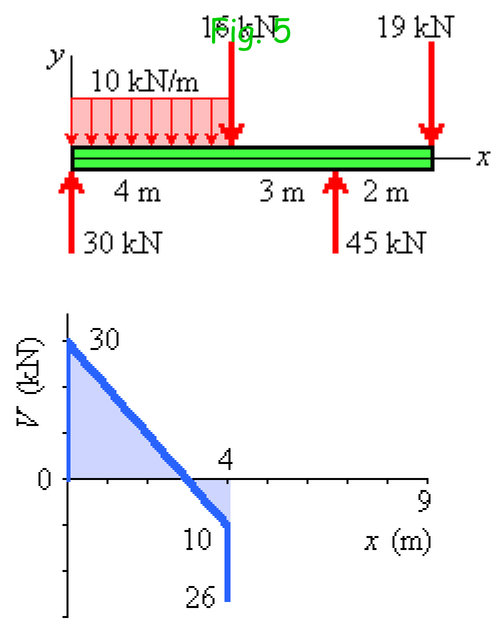
Fig. 4

We also need to know where the shear force becomes zero. We know that the full 4 m of the distributed load causes a change in the shear force of 40 kN. So how much of the distributed load will it take to cause a change of 30 kN (from +30 kN to 0 kN)? Since the distributed load is uniform, the area (change in shear force) is just $10 \times b = 30$, which gives $b = 3$ m. That is, the shear force graph becomes zero at x = 3 m (3 m from the beginning of the uniform distributed load).

Concentrated Force

The 16-kN concentrated force at *B* causes the shear force graph to jump down (in the direction of the force) by 16 kN (the magnitude of the force) from -10 kN to -26 kN.

Fig. 5



No Loads

Since there are no loads between points *B* and *C*, the shear force graph is constant (the slope $dV/dx = w = 0$) at -26 kN.

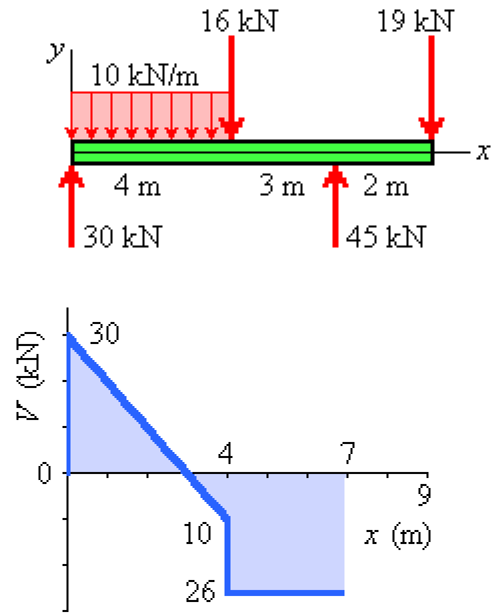


Fig. 6

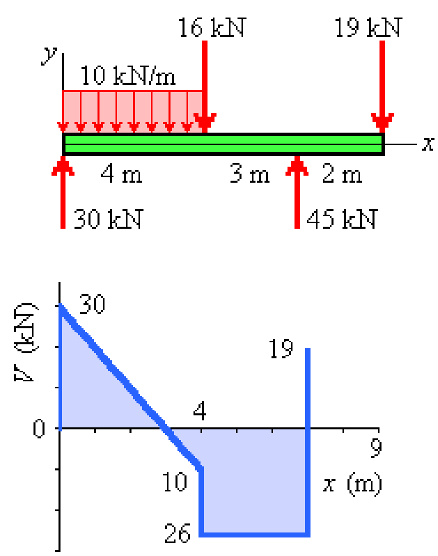


Fig. 7

Concentrated Force

The 45-kN concentrated force (support reaction) at *C* causes the shear force graph to jump up (in the direction of the force) by 45 kN (the magnitude of the force) from -26 kN to +19 kN.

No Loads

Since there are no loads between points C and D, the shear force graph is constant (the slope $dV/dx = w = 0$) at +19 kN.

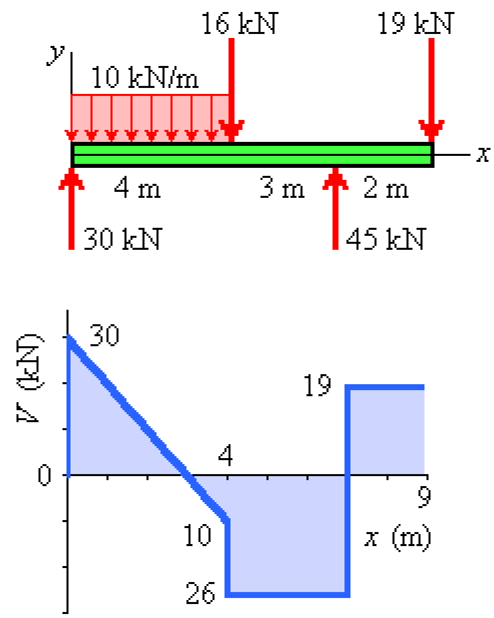


Fig. 8

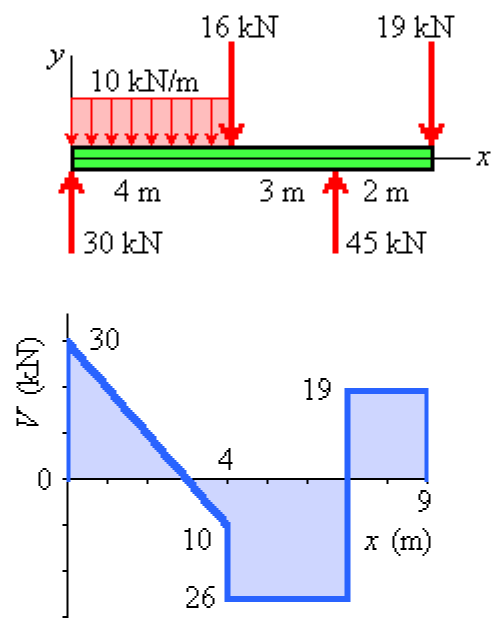


Fig. 9

Concentrated Force

The 19-kN concentrated force at *D* causes the shear force graph to jump down (in the direction of the force) by 19 kN (the magnitude of the force) from +19 kN to 0 kN.

Drawing the Bending Moment Diagram

Since there are no concentrated moments acting on this beam, the bending moment diagram (graph) will be continuous (no jumps) and it will start and end at zero.

Decreasing Shear Force

The bending moment graph starts out at zero and with a large positive slope (since the shear force starts out with a large positive value and $dM/dx = V$). As the shear force decreases, so does the slope of the bending moment graph. At $x = 3$ m the shear force becomes zero and the bending moment is at a local maximum ($dM/dx = V = 0$) For values of x greater than 3 m ($3 < x < 4$ m) the shear force is negative and the bending moment decreases ($dM/dx = V < 0$).

The shear force graph is linear (1st order function of x), so the bending moment graph is a parabola (2nd order function of x).

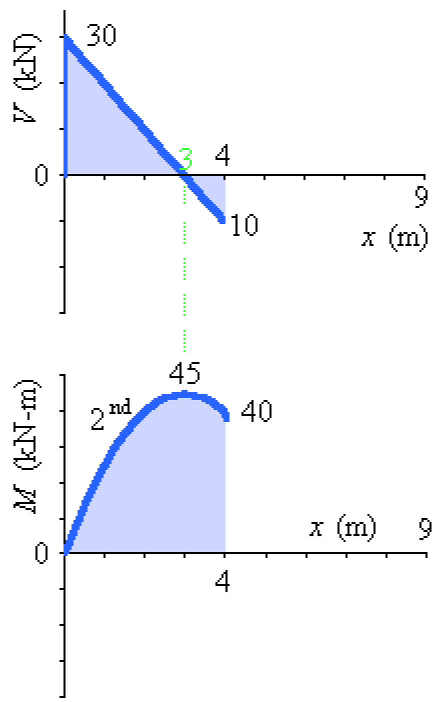


Fig. 10

The change in the bending moment between $x = 0$ m and $x = 3$ m is equal to the area under the shear graph between those two points. The area of the triangle is

$$\Delta M = (1/2)(30 \times 3) = 45 \text{ kN}\cdot\text{m}$$

So the value of the bending moment at $x = 3$ m is $M = 0 + 45 = 45$ kN·m. The change in the bending moment between $x = 3$ and $x = 4$ m is also equal to the area under the shear graph

$$\Delta M = (1/2)(-10 \times 1) = -5 \text{ kN·m}$$

So the value of the bending moment at $x = 4$ m is $M = 45 - 5 = 40$ kN·m.

Constant Shear Force

Although the bending moment graph is continuous at $x = 4$ m, the jump in the shear force at $x = 4$ m causes the slope of the bending moment to change suddenly from $dM/dx = V = -10$ kN·m/m to $dM/dx = -26$ kN·m/m.

Since the shear force graph is constant between $x = 4$ m and $x = 7$ m, the bending moment graph has a constant slope between $x = 4$ m and $x = 7$ m ($dM/dx = V = -26$ kN·m/m). That is, the bending moment graph is a straight line.

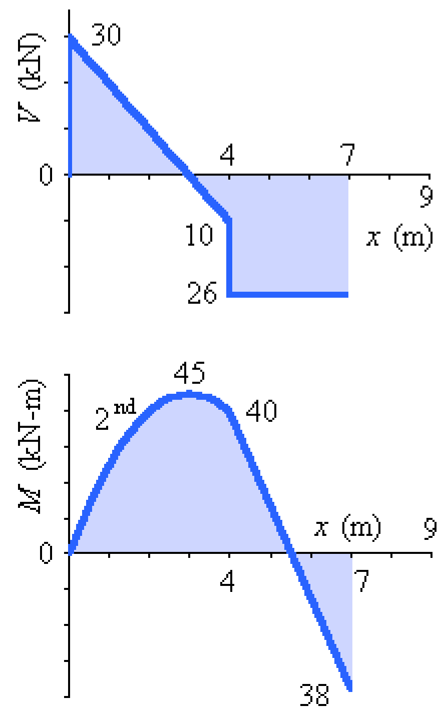


Fig. 11

The change in the bending moment between $x = 4$ m and $x = 7$ m is equal to the area under the shear graph between those two points. The area of the rectangle is just $\Delta M = (-26 \times 3) = -78$ kN·m. So the value of the bending moment at $x = 7$ m is $M = 40 - 78 = -38$ kN·m.

Constant Shear Force

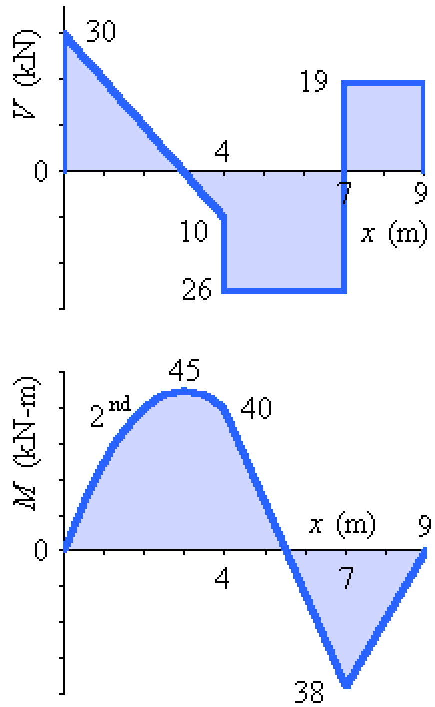


Fig. 12

Again the bending moment graph is continuous at x = 7 m. The jump in the shear force at x = 7 m causes the slope of the bending moment to change suddenly from dM/dx = V = -26 kN·m/m to dM/dx = +19 kN·m/m.

Since the shear force graph is constant between x = 7 m and x = 9 m, the bending moment graph has a constant slope between x = 7 m and x = 9 m (dM/dx = V = +19 kN·m/m). That is, the bending moment graph is a straight line.

The change in the bending moment between x = 7 m and x = 9 m is equal to the area under the shear graph between those two points. The area of the rectangle is just ΔM = (+19 × 2) = +38 kN·m. So the value of the bending moment at x = 7 m is M = -38 + 38 = 0 kN·m.