III. Addition of Angular Momenta

a. Coupled and Uncoupled Bases

When dealing with two different sources of angular momentum, $\hat{\mathbf{J}}_1$ and $\hat{\mathbf{J}}_2$, there are two obvious bases that one might choose to work in. The first is called the ***uncoupled basis.*** Here the basis kets are eigenstates of *both* operators:

$$\hat{\mathbf{J}}_1^2 \left| j_1, m_1; j_2, m_2 \right\rangle = j_1(j_1+1) \left| j_1, m_1; j_2, m_2 \right\rangle$$

$$\hat{J}_{1z} \left| j_1, m_1; j_2, m_2 \right\rangle = m_1 \left| j_1, m_1; j_2, m_2 \right\rangle$$

$$\hat{\mathbf{J}}_2^2 \left| j_1, m_1; j_2, m_2 \right\rangle = j_2(j_2+1) \left| j_1, m_1; j_2, m_2 \right\rangle$$

$$\hat{J}_{2z} \left| j_1, m_1; j_2, m_2 \right\rangle = m_2 \left| j_1, m_1; j_2, m_2 \right\rangle$$

In the case of spin orbit coupling, this would mean that our basis states would be simultaneous eigenfunctions of orbital *and* spin angular momentum, and each state would have a particular value for $\{l, m\}$ and m_s. Note that this is only possible if (as we assume):

$$\left[\hat{\mathbf{J}}_1, \hat{\mathbf{J}}_2\right] = 0$$

otherwise, the two operators would not have simultaneous eigenfunctions. This is clearly true for $\hat{\mathbf{L}}$ and $\hat{\mathbf{S}}$ since the two operators act on different spaces. This really defines what we mean by *different* angular momenta, since operators that do not commute will, in some sense, define overlapping – and thus not completely distinct – forms of angular momentum.

This basis is appropriate if $\hat{\mathbf{J}}_1$ and $\hat{\mathbf{J}}_2$ do not interact. However, when the Hamiltonian contains an interaction between these two angular momenta (such as $\hat{\mathbf{J}}_1 \cdot \hat{\mathbf{J}}_2$), the eigenstates will be mixtures of the uncoupled basis functions and this basis becomes somewhat awkward. In these cases, it is easiest to work in the ***coupled*** basis, which we now develop.

First, note that the total angular momentum is given by:

$$\hat{\mathbf{J}} = \hat{\mathbf{J}}_1 + \hat{\mathbf{J}}_2$$

It is easy to show that this is, in fact, an angular momentum (i.e. $\left[\hat{J}_x, \hat{J}_y\right] = i\hat{J}_z$). We can therefore associate two quantum numbers, j and m, with the eigenstates of total angular momentum indicating its magnitude and projection onto the z axis. The coupled basis states are eigenfunctions of the ***total*** angular momentum operator. This specifes two quantum numbers for our basis states (j and m). However, as we saw above, the uncoupled basis states were specified by four quantum numbers (j_1, j_2, m_1 and m_2) and we therefore need to specify two more quantum numbers to fully specify the coupled states. To specify these last two quantum numbers, we note that

$$\left[\hat{\mathbf{J}}^2, \hat{J}_{1z}\right] = \left[\left(\hat{\mathbf{J}}_1^2 + 2\hat{\mathbf{J}}_1 \cdot \hat{\mathbf{J}}_2 + \hat{\mathbf{J}}_2^2\right), \hat{J}_{1z}\right] = \left[2\hat{\mathbf{J}}_1 \cdot \hat{\mathbf{J}}_2, \hat{J}_{1z}\right] = 2\left[\hat{\mathbf{J}}_1, \hat{J}_{1z}\right] \cdot \hat{\mathbf{J}}_2 \neq 0$$

and similarly for $\hat{J}_{2z}$. Thus $\hat{J}_{1z}$ and $\hat{J}_{2z}$ **do not** share common eigenfunctions with $\hat{\mathbf{J}}^2$. To put it another way, to obtain a definite state of the total angular momentum, one must generally mix states with different m_1 and m_2. All thus leads to the conclusion that neither m_1 nor m_2 can be one of the other quantum numbers that specify the coupled basis.

What about j_1 and j_2? Well,

$$\left[\hat{\mathbf{J}}^2, \hat{\mathbf{J}}_1^2\right] = \left[\left(\hat{\mathbf{J}}_1^2 + 2\hat{\mathbf{J}}_1 \cdot \hat{\mathbf{J}}_2 + \hat{\mathbf{J}}_2^2\right), \hat{\mathbf{J}}_1^2\right] = \left[2\hat{\mathbf{J}}_1 \cdot \hat{\mathbf{J}}_2, \hat{\mathbf{J}}_1^2\right] = 2\left[\hat{\mathbf{J}}_1, \hat{\mathbf{J}}_1^2\right] \cdot \hat{\mathbf{J}}_2 \neq 0$$

and similarly for $\hat{\mathbf{J}}_2^2$. Further

$$\left[\hat{J}_z, \hat{\mathbf{J}}_1^2\right] = \left[\hat{J}_{1z} + \hat{J}_{2z}, \hat{\mathbf{J}}_1^2\right] = \left[\hat{J}_{1z}, \hat{\mathbf{J}}_1^2\right] \neq 0 .$$

Hence, $\hat{\mathbf{J}}^2, \hat{J}_z, \hat{\mathbf{J}}_1^2$ and $\hat{\mathbf{J}}_2^2$ share common eigenfunctions, and these eigenfunctions define the coupled basis. To say it another way, the eigenstates of $\hat{\mathbf{J}}^2$ do not mix states with different j_1 and j_2. This is rather profound – in the case of spin-orbit coupling it means that states with different values of l will not be mixed by the coupling.

The appropriate quantum numbers for the coupled basis are j, m, j_1 and j_2 and we have:

$$\hat{\mathbf{J}}^2 \left| j,m;j_1,j_2 \right\rangle = j(j+1) \left| j,m;j_1,j_2 \right\rangle$$

$$\hat{J}_z \left| j,m;j_1,j_2 \right\rangle = m \left| j,m;j_1,j_2 \right\rangle$$

$$\hat{\mathbf{J}}_1^2 \left| j,m;j_1,j_2 \right\rangle = j_1(j_1+1) \left| j,m;j_1,j_2 \right\rangle$$

$$\hat{\mathbf{J}}_2^2 \left| j,m;j_1,j_2 \right\rangle = j_2(j_2+1) \left| j,m;j_1,j_2 \right\rangle$$

Typically, certain matrix elements will be easier to compute in the coupled basis, while others will be easier to compute in the uncoupled basis. Thus, we will often need to *transform* from one basis to the other. Since the coupled and uncoupled bases are both eigenfunctions of Hermitian operators, each forms a complete basis for the angular momentum and therefore, we can write:

$$\left| j,m;j_1,j_2 \right\rangle = \sum_{\substack{j_1',m_1 \\ j_2',m_2}} \left| j_1',m_1;j_2',m_2 \right\rangle \left\langle j_1',m_1;j_2',m_2 \right| j,m;j_1,j_2 \rangle .$$

However, we have already concluded that $\hat{\mathbf{J}}^2$ does not mix states with different j_1 and j_2 so:

$$\left\langle j_1',m_1;j_2',m_2 \right| j,m;j_1,j_2 \rangle = \left\langle j_1,m_1;j_2,m_2 \right| j,m;j_1,j_2 \rangle \delta_{j_1,j_1'} \delta_{j_2,j_2'}$$

As a result the sums over j_1' and j_2' collapse to delta functions and we get:

$$\left| j,m;j_1,j_2 \right\rangle = \sum_{m_1,m_2} \left| j_1,m_1;j_2,m_2 \right\rangle \left\langle j_1,m_1;j_2,m_2 \right| j,m;j_1,j_2 \rangle$$

The transformation coefficients $\left\langle j_1,m_1;j_2,m_2 \right| j,m;j_1,j_2 \rangle$ are known as the Clebsch-Gordon (CG) coefficients (or the vector coupling coefficients). The CG matrix is unitary (since it just transforms a vector from one basis to anotherand by convention its elements are chosen real (recall that the phase of $\left| j,m;j_1,j_2 \right\rangle$ is arbitrary).

There is one additional symmetry that the CG coefficients possess. Notice that:

$$0 = \left(m - \hat{J}_z\right) \left| j,m;j_1,j_2 \right\rangle = \left(m - \hat{J}_{1z} - \hat{J}_{2z}\right) \left| j,m;j_1,j_2 \right\rangle$$

$$\Rightarrow 0 = \left\langle j_1,m_1;j_2,m_2 \right| \left(m - \hat{J}_{1z} - \hat{J}_{2z}\right) \left| j,m;j_1,j_2 \right\rangle$$

$$\Rightarrow 0 = (m - m_1 - m_2) \left\langle j_1,m_1;j_2,m_2 \right| j,m;j_1,j_2 \rangle$$

This implies that either the CG coefficient is zero, or

$$m = m_1 + m_2 .$$

Thus, for all the non-zero CG coefficients, the index m is actually redundant; it is *always* given by the sum of m_1 and m_2.

b. Recursion Relations

There are several ways to determine the CG coefficients. Perhaps the easiest is to look them up in a book (they have been extensively tabulated). However, this method is the most prone to error, unless you are very careful to follow all of the sign conventions of the text at hand (which may not be the same as the sign conventions in, say CTDL or these lecture notes).

Another route is to simply view the whole thing as an eigenvalue problem: one simply wishes to determine the eigenstates of $\hat{\mathbf{J}}^2$ in the uncoupled basis. The coefficients of the different eigenvectors are the CG coefficients. However, this misses out on what is probably the most important aspect of angular momentum coupling – the ability **without any significant computation** to predict the allowed quantum numbers and their degeneracies.

We will follow a third route to obtain the CG coefficients. This notes that the coefficients are easily obtained by recursion, in a manner similar to what we used for the spherical harmonics. First we note that there is only one non-zero coefficient for $m_{\max} = m_{1\max} + m_{2\max}$: $\langle j_1, m_{1\max}; j_2, m_{2\max} | j, m_{\max}; j_1, j_2 \rangle$. No other combination of m_1 and m_2 will give the correct total m. Thus, the states $| j, m_{\max}; j_1, j_2 \rangle$ and $| j_1, m_{1\max}; j_2, m_{2\max} \rangle$ are **equal** up to an unimportant constant. By convention, this constant is chosen to be 1. We can generate the other coefficients by successive applications of the lowering operator and judicious use of orthogonality constraints.

To see how this is applied in practice, it is best to use spin-orbit coupling as an example. What is the coupled basis for a spin-1/2 electron with orbital angular momentum l? We can identify the uncoupled quantum numbers:

$$j_1 = l \qquad m_1 = m_l$$

$$j_2 = s = \tfrac{1}{2} \qquad m_2 = m_s$$

And we want to determine the eigenstates of $\hat{\mathbf{J}}^2 \equiv \left(\hat{\mathbf{L}} + \hat{\mathbf{S}}\right)^2$ and $\hat{J}_z = \hat{L}_z + \hat{S}_z$. We know now that the two states with maximum m are equal:

$$\underline{\text{Coupled:}} \qquad \underline{\text{Uncoupled:}}$$

$$\left|l+\tfrac{1}{2}, m = l+\tfrac{1}{2}; l, s\right\rangle = \left|l, m_l = l; s, +\tfrac{1}{2}\right\rangle.$$

Applying the lowering operator, we have

$$\hat{J}_- \left|l+\tfrac{1}{2}, m = l+\tfrac{1}{2}; l, s\right\rangle = \left(\hat{J}_{l-} + \hat{J}_{s-}\right)\left|l, m_l = l; s, +\tfrac{1}{2}\right\rangle$$

$$= \sqrt{(l+l)(l-l+1)}\left|l, l-1; s, +\tfrac{1}{2}\right\rangle + \sqrt{\left(s+\tfrac{1}{2}\right)\left(s-\tfrac{1}{2}+1\right)}\left|l, l; s, -\tfrac{1}{2}\right\rangle$$

$$= \sqrt{2l}\left|l, l-1; s, +\tfrac{1}{2}\right\rangle + \left|l, l; s, -\tfrac{1}{2}\right\rangle$$

However, we also know that

$$\hat{J}_- \left|l+\tfrac{1}{2}, m = l+\tfrac{1}{2}; l, s\right\rangle = \sqrt{\left(l+\tfrac{1}{2}+l+\tfrac{1}{2}\right)\left(l+\tfrac{1}{2}-l-\tfrac{1}{2}+1\right)}\left|l+\tfrac{1}{2}, m = l-\tfrac{1}{2}; l, s\right\rangle$$

$$= \sqrt{2l+1}\left|l+\tfrac{1}{2}, m = l-\tfrac{1}{2}; l, s\right\rangle$$

Combining these last two expressions,

$$\left|l+\tfrac{1}{2}, m = l-\tfrac{1}{2}; l, s\right\rangle = \sqrt{\frac{2l}{2l+1}}\left|l, l-1; s, +\tfrac{1}{2}\right\rangle + \sqrt{\frac{1}{2l+1}}\left|l, l; s, -\tfrac{1}{2}\right\rangle$$

This gives us the expression for the state with the same j, l and s but $m = m_{\max} - 1$. One can check that this state is normalized. We can clearly apply this recursively to obtain the states with $m = m_{\max} - 2$, $m = m_{\max} - 3$, etc. Clearly this will cease when $m = -l - \tfrac{1}{2} = m_{\min}$.

To get some insight into what these states look like, we need to make our notation a little less explicit, temporarily. First we will delete the indices for l and s from all the bra and ket states since these quantum numbers are the same throughout the calculation. So, for example, $\left|j, m; l, s\right\rangle \to \left|j, m\right\rangle$ and $\left|l, m_l; s, m_s\right\rangle \to \left|m_l; m_s\right\rangle$ Second, in terms of the abbreviated state labels, we will write the above relationship symbolically

$$\left|l+\tfrac{1}{2}, l-\tfrac{1}{2}\right\rangle \approx \left|l+\tfrac{1}{2}, l-\tfrac{1}{2}\right\rangle \approx \left|l-1; +\tfrac{1}{2}\right\rangle + \left|l; -\tfrac{1}{2}\right\rangle$$

where $\approx$ means roughly “neglecting any constants that are not relevant for the point I want to make”. In this case, they are factors

involving l and s that will be very important for doing calculations but impede our understanding at the outset.

Using this notation, we can write the next lowered state:

$$\left|l+\tfrac{1}{2},l-\tfrac{3}{2}\right\rangle \approx \hat{J}_{-}\left|l+\tfrac{1}{2},l-\tfrac{1}{2}\right\rangle \approx \left(\hat{J}_{l-}+\hat{J}_{s-}\right)\left|l-1;+\tfrac{1}{2}\right\rangle + \left(\hat{J}_{l-}+\hat{J}_{s-}\right)\left|l;-\tfrac{1}{2}\right\rangle$$

$$\approx \qquad \left|l-2;+\tfrac{1}{2}\right\rangle \quad + \quad \left|l-1;-\tfrac{1}{2}\right\rangle$$

In fact, if we make rows of each of the coupled states we can make a flow chart for the different components:

$$\left|l+\tfrac{1}{2},l+\tfrac{1}{2}\right\rangle \approx \left|l;+\tfrac{1}{2}\right\rangle$$

$$\left|l+\tfrac{1}{2},l-\tfrac{1}{2}\right\rangle \approx \left|l-1;+\tfrac{1}{2}\right\rangle + \left|l;-\tfrac{1}{2}\right\rangle$$

$$\left|l+\tfrac{1}{2},l-\tfrac{3}{2}\right\rangle \approx \left|l-2;+\tfrac{1}{2}\right\rangle + \left|l-1;-\tfrac{1}{2}\right\rangle$$

...

$$\left|l+\tfrac{1}{2},-l+\tfrac{1}{2}\right\rangle \approx \left|-l;+\tfrac{1}{2}\right\rangle + \left|-l+1;-\tfrac{1}{2}\right\rangle$$

$$\left|l+\tfrac{1}{2},l-\tfrac{1}{2}\right\rangle \approx \qquad \left|-l;-\tfrac{1}{2}\right\rangle$$

So we see that the characteristic action of the lowering operator is to connect upper states in (m_l, m_s) space to states that are below and to the right of the original state. This action is limited by the fact that m_l cannot be less than $-l$ (this determines the "height" of the ladder) and m_s cannot be less than $-\frac{1}{2}$ (this determines the widths of the rungs).

c. <u>The triangle rule</u>

So, have we now created all the coupled states? Well, the total number of coupled states in the ladder is $2j+1 = 2\left(l+\frac{1}{2}\right) = 2l+2$.

Meanwhile, the number of uncoupled states is $(2l+1)(2s+1)=(2l+1)2=4l+2$. Since the number of coupled and uncoupled states **must** be equal, we are missing $2l$ states. To find the missing states, notice that all of the above states have $j=l+\frac{1}{2}$. That is, we have presumed that l and s point in the same direction (equivalently, we have presumed that they are rotating in the same plane an in the same sense – clockwise or counterclockwise). This is clearly an unnecessary restriction. Hence, we expect there to be another possibility for j - specifically, in order to account for all the "missing" states, we expect $2j+1=2l$, or $j=l-\frac{1}{2}$.

To build these missing states, we note that there are two uncoupled functions with $m=m_{\max}-1$: $\left|m_l=l-1;m_s=\frac{1}{2}\right\rangle$ and $\left|m_l=l;m_s=-\frac{1}{2}\right\rangle$. Meanwhile, we have only found one coupled state $\left|m_l=l-1;m_s+\frac{1}{2}\right\rangle$: $\left|j=l+\frac{1}{2};m=l-\frac{1}{2}\right\rangle$. Based on the above arguments, we predict there will also be a state $\left|j=l-\frac{1}{2};m=l-\frac{1}{2}\right\rangle$. To find it we note that 1) this state must be a linear combination of $\left|m_l=l-1;m_s=\frac{1}{2}\right\rangle$ and $\left|m_l=l;m_s=-\frac{1}{2}\right\rangle$, since other states do not conserve m 2) the new state must be orthogonal to the $\left|j=l+\frac{1}{2};m=l-\frac{1}{2}\right\rangle$, since both states are eigenstates of the same operator ($\hat{\mathbf{J}}^2$). Using our explicit expression for $\left|j=l+\frac{1}{2};m=l-\frac{1}{2}\right\rangle$ (above), it is easy to show that the normalized state satisfying 1) and 2) is:

$$\left|l-\tfrac{1}{2},m=l-\tfrac{1}{2};l,s\right\rangle=\sqrt{\frac{1}{2l+1}}\left|l,l-1;s,+\tfrac{1}{2}\right\rangle-\sqrt{\frac{2l}{2l+1}}\left|l,l;s,-\tfrac{1}{2}\right\rangle$$

where we have once again made an arbitrary choice of phase, so that the coefficient of the first term is positive. Starting from this state, we can apply the lowering operator recursively to generate the $2l$ states $\left|l-\frac{1}{2},m;l,s\right\rangle$ with all other possible values of m.

How do these ladders generalize to arbitrary m_1 and m_2? Well, one finds that the possible values of j fall between two limits:

$$\left|j_1-j_2\right|\le j\le j_1+j_2$$

This may be familiar to some of you as the "triangle rule" of angular momentum coupling. Physically, this comes from the fact that the maximum number of states we can have with a given m is $2j_2+1$

(assuming $j_1 > j_2$). Pictorially, this corresponds to the maximum width of the "rungs" on our ladder. Now, when we write out a table of the m values for each j, we find:

	$j=j_1+j_2$	$j=j_1+j_2-1$	$j=j_1+j_2-2$	$j=j_1+j_2-3$
$m= j_1+j_2$	X			
$m=j_1+j_2-1$	X	X		
$m=j_1+j_2-2$	X	X	X	
$m=j_1+j_2-3$	X	X	X	X
…	X	X	X	X
$m=-j_1-j_2+3$	X	X	X	X
$m=-j_1-j_2+2$	X	X	X	
$m=-j_1-j_2+1$	X	X		
$m=-j_1-j_2$	X			

We see that having n different values for j requires us to have n different states with a given value of m. For example, in the chart above, we would need four states with $m=j_1+j_2-3$ to support the four values of j that are listed. Since the maximum degeneracy of each m level is $2j_2+1$, there are $2j_2+1$ possible values of j and we conclude that they are $|j_1 - j_2| \le j \le j_1 + j_2$.

We can also check that the number of states predicted by the triangle rule agrees with what we know from the uncoupled basis. For each value of j we have $2j+1$ states, so the total number of states is (if we assume that $j_1 > j_2$):

$$(2(j_1+j_2)+1)+(2(j_1+j_2-1)+1)+\ldots+(2(j_1-j_2+1)+1)+(2(j_1-j_2)+1).$$

Or, rearranging the terms,

$$(2j_1+1)+2j_2+(2j_1+1)+2j_2-2+\ldots+(2j_1+1)-2j_2+2+(2j_1+1)+2j_2.$$

The terms outside the parentheses clearly sum to zero. Meanwhile, there are exactly $2j_2+1$ copies of the term in parentheses. Hence, the number of states is

$$(2j_1+1)(2j_2+1)$$

which is precisely the number of states in the uncoupled representation.