MASSACHUSETTS INSTITUTE OF TECHNOLOGY

5.73 Quantum Mechanics I
Fall, 2003

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Problem Set #1

DUE: At the start of Lecture on Friday, September 12.

Reading: C-TDL, pages 9-39, 50-56, 60-85

Problems:

1. Figure 1 (p. 84) of Complement J_I of C-TDL shows $|\Psi(x,t)|^2$ for a Gaussian wavepacket scattering off a step barrier. A Gaussian in x has the form (see handout)

$$G(x;x_0,\Delta x) = (2\pi)^{-1/2}\frac{1}{\Delta x}e^{-(x-x_0)^2/[2(\Delta x)^2]}$$

(is this normalized to 1 in the sense $\int_{-\pi_0}^{\infty} dx\, G(x) = 1$ or $\int dx\, |G(x)|^2 = 1$?) where x_0 is the center and $(\Delta x)^2$ is the variance ("width"). It is convenient to start with a minimum uncertainty wavepacket at $t = 0$,

$$\Delta x \Delta k = 1/2.$$

Since, for a minimum uncertainty wavepacket subject to a constant potential energy, V_0,

$$E - V_0 = p^2/2m = \frac{\hbar}{2m}k^2,$$

specification of Δx determines Δk (hence ΔE). To make the wavepacket move we need to convert it to a Gaussian wavepacket in k (so that the E dependence becomes explicit) with center located at t = 0 at $x = x_0$. The form of the wavepacket in k is

$$\Psi(x,t) = \int dk\, g(k) e^{i\alpha(k)} e^{ikx} e^{-iE(k)t}$$

where g(k) is a Gaussian in k (centered at k_0 and with $\Delta k = 1/2\Delta x$) and $\alpha(k)$ is designed so that, at t = 0, the center of the wavepacket is located at x_0. So now you are, in principle, equipped to create a wavepacket with specified t = 0 values of E, ΔE, x_0, Δx, k_0, and Δk. THIS IS AN OPEN-ENDED PROBLEM. You could answer the following questions by making a computer movie of $|\Psi(x,t)|^2$ or you could attempt to extract the necessary insights from the E-dependent solutions of the time-independent Schrödinger equation

$$\mathbf{H}\psi_E(x) = E\psi_E(x).$$

It is not difficult to obtain an analytic solution for $\psi_E(x)$ at all E.

A. What is happening in part C of Figure 1? What are the fringes at $x < 0$? Why are there no fringes at $x > 0$? What change in E and/or m would cause the fringe spacing to double? What controls the modulation depth (define this any way you think reasonable) of the fringes?

B. What is happening in part D of the figure? What controls the relative areas of the two peaks? Are there any resonances in the ratio of the areas of the transmitted and reflected peaks? If there are no resonances, why not? If there is a resonance, what feature of the potential is responsible for it?

C. The left and right peaks in part D have different widths and are centered at different distances from $x = 0$. Why?

D. Now this is the hard part. Consider a potential of the form

$$V(x) = V_1 > 0 \quad x < 0$$
$$V(x) = 0 \quad 0 \le x \le a$$
$$V(x) = V_2 > V_1 \quad a < x.$$

Devise a series of wavepacket experiments where you determine V_1, V_2, and *a* uniquely. Explain what features in the wavepacket scattering experiment are especially sensitive to each of the unknown system parameters. Explain how to choose a set of experimental conditions so that the set of measurements is most sensitive to V_1, V_2, and *a*. You are allowed to devise separate experiments in which the wavepacket is incident from the right and from the left.

E. In your answers to the questions in part D, you probably used direct observations of the form of $\Psi(x,t)$ or $|\Psi(x,t)|$ at various times in its traversal of the steps in the potential. Of course, this is illegal. One can never observe the wavefunction itself. Under certain circumstances $|\Psi(x,t)|^2$ is observable, not over all x and t, but at specific values of x for all t (or specific values of t for all x, but such a measurement requires a "sophisticated" experiment). One usually measures flux arriving from the right at $x \ll 0$ or from the left at $x \gg a$. Now discuss how your experiments would sample V_1, V_2, and *a* subject to these limitations.

2. C-TDL, page 86, #2.

3. C-TDL, page 88, #5.