Vector Calculus Independent Study

Unit 2: Parametric Curves

In this unit you should/will learn:

1. The parametric curve $\vec{\sigma}(t) = (x(t), y(t), z(t))$ describes a path through space with location $(x(t), y(t), z(t))$ at time t.

2. The derivative $\frac{d}{dt}\vec{\sigma}(t) = \vec{\sigma}'(t) = (x'(t), y'(t), z'(t))$ of the path gives the velocity vector at time t. Any multiple of the velocity vector is tangent to the curve, so it is useful to define the unit tangent vector $\vec{T} = \vec{\sigma}'(t)/|\vec{\sigma}'(t)|$. Speed is defined to be the magnitude of velocity, and is thus $|\vec{\sigma}'(t)|$. Similarly, the accleration vector is given by $\vec{\sigma}''(t)$.

3. The arc length of the curve can be obtained by integrating its speed. Thus, $\int_a^b |\vec{\sigma}'(t)|\,dt$ gives the length along the curve from $\vec{\sigma}(a)$ to $\vec{\sigma}(b)$.

4. The path integral of f over the curve $\vec{\sigma}$ is defined as

$$\int_{\vec{\sigma}} f\,ds = \int f|\vec{\sigma}'(t)|dt$$

   ds can be thought of as an infinitesmial increase in arc length, so $ds = |\vec{\sigma}'(t)|dt$.

5. The path integral $\int_{\vec{\sigma}} f\,ds$ has many applications, including:

   - The surface area of a fence of height f over a base $\vec{\sigma}$, assuming $\vec{\sigma}$ is contained in the plane.
   - The arc length of $\vec{\sigma}$, assuming $f = 1$.
   - The average value of f over the curve.
   - The mass of the wire $\vec{\sigma}$, assuming f gives a density in terms of mass per unit length.

6. If $g(t)$ is an increasing continuous function, then $\vec{\sigma}(g(t))$ traces out the same geometric curve as $\vec{\sigma}(t)$ and is said to be a reparameterization of $\vec{\sigma}(t)$. A reparameterization can have velocity and accleration vectors that are very different from those of the original curve, but it will have the same length and path integrals.

7. The work done by moving a particle over a vector displacement $\vec{v}$ through a constant force $\vec{F}$ is given by the dot product $\vec{v} \cdot \vec{F}$. The work done by moving a particle over a path $\vec{\sigma}$ through a location dependendent force field $\vec{F}(x, y, z)$ is given by the work integral of $\vec{\sigma}$ through the force field $\vec{F}$,

$$\int_{\vec{\sigma}} \vec{F} \cdot d\vec{s} = \int \vec{F}(\vec{\sigma}(t)) \cdot \vec{\sigma}'(t)\, dt$$

$d\vec{s}$ can be thought of as an infinitesmial velocity vector, as in $d\vec{s} = \vec{\sigma}'(t)\, dt$.

Suggested Procedure:

1. Read and do some problems from
   - Rogers Chapters 6, 7 and 20,
   - Marsden and Tromba 3rd edition sections 3.1, 3.2, 7.1, and 7.2,
   - Marsden and Tromba 4th edition sections 2.4, 4.1, 4.2, 7.1, and 7.2,
   - Thomas & Finney chapter 13 and 14, or
   - Simmons chapter 17, and section 21.1
2. Take the sample test.
3. Take a unit test.