Vector Calculus Independent Study

Unit 4: Maximization and Minimization

To find the maximium and minimum points of a function $f(\vec{x})$ on a set D, follow this procedure:

1. Look at all the *critical points* $\vec{x}$ where $\nabla f(\vec{x}) = 0$. Throw out any that aren't in D.
2. Look at any points where f is non-differentiable.
3. Look at the boundary of D. You can do this either by
   - Parameterizing the boundary so that you have an unconstrained max/min problem, or by
   - Lagrange multipliers: if the boundary of D is described as a level set by $g(\vec{x}) = 0$, then the critical points of f constrained to be on the boundary of D can be found by solving $\nabla f = \lambda \nabla g$.
4. If $\nabla h(\vec{x}) = 0$, then you can tell if $\vec{x}$ is a local max, min, or saddle point by looking at the Hessian:

$$H(f) = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1} & \dfrac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

   If all of the upper-left square sub-matrices of the Hessian have positive determinant, then the critical point is a local minimum. If the determinants alternate in sign starting with $\frac{\partial^2 f}{\partial x_1^2} < 0$, then the critical point is a local maximum. If any of the determinants are zero, then the test is indeterminant. Finally, if none of the above ifs apply, you have a saddle point.

Suggested Procedure:

1. Read and do some problems from
    - Rogers Chapters 12 and 13,
    - Marsden and Tromba 3rd edition chapter 4,
    - Marsden and Tromba 4th edition chapter 3, or
    - Simmons, sections 19.7 and 19.8.
2. Take the sample test.
3. Take a unit test.