Vector Calculus Independent Study

Unit 5: Vector Fields

A vector field is a function which associates a vector to every point in space. Vector fields are everywhere in nature, from the wind (which has a velocity vector at every point) to gravity (which, in the simplest interpretation, would exert a vector force at on a mass at every point) to the gradient of any scalar field (for example, the gradient of the temperature field assigns to each point a vector which says which direction to travel if you want to get hotter fastest).

In this section, you will learn the following techniques and topics:

- How to graph a vector field by picking lots of points, evaluating the field at those points, and then drawing the resulting vector with its tail at the point.

- A flow line for a velocity vector field is a path $\vec{\sigma}(t)$ that satisfies

  $$\vec{\sigma}'(t) = \vec{F}(\vec{\sigma}(t))$$

  For example, a tiny speck of dust in the wind follows a flow line. If you have an acceleration vector field, a flow line path satisfies

  $$\vec{\sigma}''(t) = \vec{F}(\vec{\sigma}(t))$$

  [For example, a tiny comet being acted on by gravity.]

- Any vector field $\vec{F}$ which is equal to ∇f for some f is called a *conservative* vector field, and f its *potential*. The terminology comes from physics; by the **fundamental theorem of calculus for work integrals**, the work done by moving from one point to another in a conservative vector field doesn't depend on the path and is simply the difference in potential at the two points.

- There are two fundamental derivatives we can take of a vector field. One is the divergence,

  $$\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

As you can see, the divergence of a vector field is a scalar field; this scalar field measures how much "stuff" is being created or destroyed in the vector field.

The other fundamental derivative of a vector field is the curl,

$$\operatorname{curl} \vec{F} = \nabla \times \vec{F} = (\frac{\partial F_2}{\partial z} - \frac{\partial F_3}{\partial y})\,\mathbf{i} - (\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x})\,\mathbf{j} + (\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x})\,\mathbf{k}.$$

The curl of a vector field is another vector field. The direction of the curl tells you what axis the vector field is rotating around, and the magnitude tells you how fast it is rotating.

- There are two relationships between all these derivatives that you need to know. First,

$$\nabla \times (\nabla f) = \vec{0}$$

or *The curl of a conservative field is the zero vector.* I like stating this theorem as "There are no Escher staircases" because it says that you can't be continuously walking up (following a gradient field) and going around in a circle (having non-zero curl) at the same time.

The other relationship is

$$\nabla \cdot (\nabla \times \vec{F}) = 0$$

or *The divergence of a curl field is zero.* The proofs of this statement and the one before it are both easy, and rely on the equality of mixed partial derivatives.

Suggested Procedure:

1. Read and do some problems from
   - Rogers Chapters 19 and 21,
   - Marsden and Tromba third edition sections 3.3, 3.4, 3.5, and 8.3, or
   - Marsden and Tromba fourth edition sections 4.3, 4.4, and 8.3.
2. Take the sample test.
3. Take a unit test.