Vector Calculus Independent Study

Unit 6: Double and Triple Integrals

Single integrals, the integrals you learned all about in calculus, find the area under the graph of a function of one variable. Double integrals, the integrals you will learn about in this section, find the volume under the graph of a function of two variables. How do you calculate a double integral? You take two integrals and call me in the morning. No, seriously, that's what you do.

And triple integrals? Well, they find the hyper-volume under the graph of a function of three variables. I mean, duh.

In this section, you will learn:

- Cavaleri's Principle: The volume of a shape which has cross-sectional area $A(x)$ [i.e., the area of the shape intersected with the plane $x = c$ is $A(c)$] can be found by $V = \int A(x)\,dx$.

- The definition of the double integral $\iint\limits_R f(x,y)\,dA$ as the limit of Riemann sums which approximate the volume under the graph of $f(x,y)$ over the planar region R.

- Double integrals are iterated integrals:

  $$\iint f(x,y)\,dA = \iint f(x,y)\,dx\,dy = \int\Big[\int f(x,y)\,dx\Big]\,dy$$

  Specifically, $\int_a^b \int_c^d f(x,y)\,dx\,dy$ integrates f over the *rectangle* $a \le y \le b$, $c \le x \le d$. $\int_a^b \int_{g(y)}^{h(y)} f(x,y)\,dx\,dy$ integrates f over the shape with $a \le y \le b$, $g(y) \le x \le h(y)$. To integrate over a region $dx\,dy$ first find the absolute max and min values of y over the region. These are the upper and lower bounds of your first integral. Then find the max and min value of x in your region *as a function of* y.

- Fubini's Theorem:

  $$\iint\limits_R f(x,y)\,dx\,dy = \iint\limits_R tf(x,y)\,dy\,dx$$

- Changing the order of integration can either be done geometrically (graph the region!) or algebraically (write down the constraints, then manipulate them).

- All of the above is true of triple integrals as well, except that you have one more variable. $\int! \int! \int 1\,dV$ is the volume of the region being integrated over. An arbitrary triple integral with limits looks like

$$\int_a^b \int_{g(z)}^{h(z)} \int_{r(y,z)}^{s(y,z)} f(x,y,z)\,dx\,dy\,dz$$

  which integrates f over the region $a \leq z \leq b$, $g(z) \leq y \leq h(z)$, $r(y,z) \leq x \leq s(y,z)$.

- Change of variables. When changing an integral $\iint\limits_R f(x,y)\,dx\,dy$ to new variables $u = g(x,y)$, $v = h(x,y)$, first change the function f to use the new variables. Next, implicitly differentiate your conversion formulas to get

$$du = \frac{\partial g}{\partial x} dx + \frac{\partial g}{\partial y} dy,$$

  etc. To get an expression $du\,dv$, just multiply du by dv and rememeber that

  1. $dx\,dx = 0$, and
  2. $dx\,dy = -dy\,dx$.

  Finally, translate the region you are integrating over to the new variables.

- Special cases:

$$\begin{aligned} \text{polar}: &\quad dx\,dy &&\to r\,dr\,d\theta \\ \text{cylindrical}: &\quad dx\,dy\,dz &&\to r\,dr\,d\theta\,dz \\ \text{spherical}: &\quad dx\,dy\,dz &&\to \rho^2 \sin\phi\,d\theta\,d\phi\,d\rho \end{aligned}$$

  By now you should certainly know that $r = \sqrt{x^2 + y^2}$, $x = \rho \sin\phi \cos\theta$, etc.; however, you should also be aware that different texts use different notation, and physics texts switch the angles in polar coordinates. You should always check to make sure that you are using the appropriate names for the coordinate variable.

- Applications. All of the below applications work for double and triple integrals, as well as for path integrals. (That is, they are all phrased in terms of double integrals, but the application is still valid if you change it to a double integral or a path integral.) In all of them, ρ is a density function with proper units (i.e., mass per area, mass per volume, or mass per length).

  1. Mass $= \iint \rho \, dA$
  2. x- coordinate of center of mass:

  $$\bar{x} = \frac{\iint x\rho \, dA}{\text{Mass}}$$

  3. Average value of f over region:

  $$\bar{f} = \frac{\iint f(x,y) \, dA}{\iint 1 \, dA}.$$

  4. Moment of inertia about the x-axis:

  $$I_x = \iint (y^2 + z^2)\rho \, dA.$$

Suggested Procedure:

1. Read and do some problems from

   - Rogers Chapters 14 - 18,
   - Marsden and Tromba chapters 5 and 6, or
   - Simmons, chapter 20.

2. Take the sample test.

3. Take a unit test.