

Vector Calculus Independent Study

Unit 7: Surfaces

We studied surfaces before when we investigated the graphs of scalar valued functions of two variables. In this unit, we generalize to the notion of parameterized surfaces. Parameterized surfaces are defined by a mapping from two space to three space – think of the straight longitude and latitude “lines” on a flat map being sent to the curved longitude and latitude lines on a sphere. While parameterized surfaces are a bit awkward at first, they soon allow you to do all sorts of things, including integrating scalar functions and vector fields over the surface.

In this section, you will learn:

- How to find the normal vector to an implicitly defined surface

$$f(x, y, z) = 0.$$

- How to describe a surface in \mathbf{R}^3 parametrically by a vector value function of two variables over some domain D .
- How to convert a surface from an implicit definition to a parametric definition. [Pick two variables and make them your parameters. Solve for the rest.]
- How to convert a surface from a parametric definition to an implicit definition. [Just eliminate the parameters. You will go from three equations in the five unknowns (u, v, x, y, z) to one equation in three unknowns (hopefully $x, y,$ and $z!$)]
- If you hold one of the parameters of a surface constant, you will define a curve on the surface. If you then take a derivative of the remaining parameter, you will get a tangent vector to the curve and thus a tangent vector to the surface. If the surface is $\vec{S}(u, v)$, then we call these two tangent vectors $\frac{\partial \vec{S}}{\partial u}$ and $\frac{\partial \vec{S}}{\partial v}$. [Marsden and Tromba likes to call them \vec{S}_u and \vec{S}_v .]

- These two vectors have another special property. If we think of a rectangle in the (u, v) -plane having its lower left hand corner at (u, v) and side lengths given by du and dv , then when we map it into \mathbf{R}^3 with \vec{S} , it will become a parallelogram with side vectors $\frac{\partial \vec{S}}{\partial u} du$ and $\frac{\partial \vec{S}}{\partial v} dv$.
- We can use this property to get a formula for the area of a surface. When we do a double integral $\iint du dv$, we are taking a sum over lots of little rectangles with infinitesimal side lengths du and dv . If we integrate $\iint \left| \frac{\partial \vec{S}}{\partial u} \times \frac{\partial \vec{S}}{\partial v} \right| du dv$, we sum over the areas (= magnitude of cross product of the side lengths) of lots of little parallelograms, all of them tangent to the surface. When they are infinitesimally small, they do a damn good job of approximating the surface area.
- Similarly, we can define $\iint f dS$, the surface integral of the function $f(x, y, z)$ over the surface \vec{S} , by the double integral

$$\iint f(\vec{S}(u, v)) \left| \frac{\partial \vec{S}}{\partial u} \times \frac{\partial \vec{S}}{\partial v} \right| du dv$$

The surface integral can be used for all of the things double, triple, and path integrals can be used for: mass, average values, moment of inertia, center of mass, etc. The surface integral of 1, $\iint 1 dS$, reduces down to our surface area formula.

- The final type of integral over a surface is the flux integral, $\iint \vec{F} \cdot d\vec{S}$. A flux integral measures how much of the vector field goes through the surface \vec{S} . One formula for the flux integral is

$$\iint \vec{F} \cdot d\vec{S} = \iint \vec{F}(\vec{S}(u, v)) \cdot \left(\frac{\partial \vec{S}}{\partial u} \times \frac{\partial \vec{S}}{\partial v} \right) du dv$$

Another equivalent formula is $\iint \vec{F} \cdot d\vec{S} = \iint \vec{F} \cdot \vec{n} dS$, where \vec{n} is a unit normal to the surface.

Suggested Procedure:

1. Read and do some problems from
 - Rogers Chapters 23 and 24, or
 - Marsden and Tromba sections 7.3, 7.4, 7.5, and 7.6.
2. Take the sample test.
3. Take a unit test.