Vector Calculus Independent Study

Unit 8: Fundamental Theorems of Vector Calculus

In single variable calculus, the fundamental theorem of calculus related the integral of the derivative of a function over an interval to the values of that function on the endpoints of the interval. In this unit, we will examine two theorems which do the same sort of thing. Gauss' theorem relates the integral of the divergence of a vector field over a solid region to the integral of the vector field over the boundary of the region, and Stokes' theorem relates the integral of the curl of a vector field over a surface to the integral of the vector field around the boundary of the surface.

In this section, you will learn:

- **Gauss' Theorem**

  $$\iiint_R \nabla \cdot \vec{F}\, dV = \iint_{\partial R} \vec{F} \cdot d\vec{S}$$

  "The triple integral of the divergence of a vector field over a region is the same as the flux of the vector field over the boundary of the region."

  Important consequences of Gauss' theorem include:

  1. The flux integral of a divergenceless vector field (i.e., a vector field $\vec{F}$ such that $\nabla \cdot \vec{F} = 0$) over a closed surface is 0.
  2. You rarely want to do a flux integral over the six sides of a cube – it is easier to do a triple integral over the inside.
  3. If the region between two closed surfaces is divergenceless, then the flux over the two surfaces is the same. (In physics, this means that two surfaces which contain the same charges have the same electromagnetic flux).
  4. The flux integral of the vector field $\vec{F} = (x, 0, 0)$ over the boundary of a region gives the volume of that region.

- **Stokes' Theorem**

$$\iint_S \nabla \times \vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{s}$$

"The flux integral of the curl of a vector field over a surface is the same as the work integral of the vector field around the boundary of the surface (just as long as the normal vector of the surface and the direction we go around the boundary agree with the right hand rule)."

Important consequences of Stokes' Theorem:

1. The flux integral of a curl field over a closed surface is 0. Why? Because it is equal to a work integral over its boundary by Stokes' Theorem, and a closed surface has no boundary!
2. Green's Theorem (aka, Stokes' Theorem in the plane): If my surface lies entirely in the plane, I can write:

$$\begin{aligned} \iint_S \nabla \times \vec{F} \cdot d\vec{S} &= \iint_S \nabla \times \vec{F} \cdot \vec{k}\, dS \\ &= \iint_S \left(\frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x} \right) dx\, dy \\ &= \int_{\partial S} (F_1, F_2) \cdot d\vec{s} \\ &= \int_{\partial S} F_1\, dx + F_2\, dy \end{aligned}$$

3. The work integral of $\int y dx$ over the boundary of a planar region R gives the area of R. (Proof: Green's Theorem).
4. The work done by going around a loop is 0 IF (a) we can make the loop into the boundary of a surface and (b) the field has curl $\vec{0}$ on the surface. This generalizes what we knew from the FTC for closed loops about conservative fields because conservative fields always have curl $\vec{0}$. Note that not all loops can be made into the boundary of a surface: think about knots with the ends put together to make a closed loop...

Suggested Procedure:

1. Read and do some problems from
   - Rogers Chapters 21 - 26, or
   - Marsden and Tromba chapter 8.
2. Take the sample test.
3. Take a unit test.