Topics in Reinforcement Learning: AlphaZero, ChatGPT, Neuro-Dynamic Programming, Model Predictive Control, Discrete Optimization
Arizona State University
Course CSE 691, Spring 2024

Links to Textbooks, Videolectures, and Slides at
http://web.mit.edu/dimitrib/www/RLbook.html

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Lecture 3
Linear Quadratic Problems, Approximation in Value Space, VI, and PI
Visualizations and Newton's Method
Problem Formulations, Reformulations, and Examples

Outline

Min-Bellman/Riccati Eqs. and L-Bellman/Riccati Eqs.

- System and cost function:

$$x_{k+1} = ax_k + bu_k, \qquad \lim_{N\to\infty} \sum_{k=0}^{N-1} (qx_k^2 + ru_k^2)$$

- The min-Bellman eq. is

$$J(x) = \min_u \left[qx^2 + ru^2 + J(ax + bu) \right]$$

It can be solved for $J^*(x)$.

- For linear $\mu(x) = Lx$, the L-Bellman eq. is

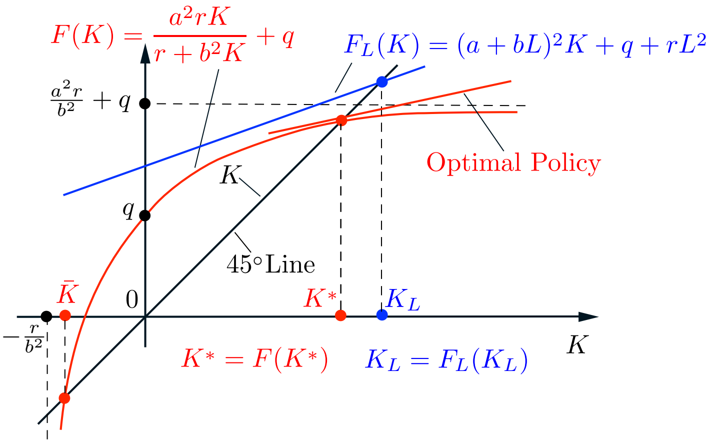
$$J(x) = (q + rL^2)x^2 + J\big((a + bL)x\big)$$

It can be solved for $J_\mu(x)$.

- We try quadratic solutions, $J(x) = Kx^2$, to Bellman eqs. and obtain the min-Riccati and L-Riccati eqs. (after we cancel x^2)

$$K = F(K) = \frac{a^2 rK}{r + b^2 K} + q, \qquad K = F_L(K) = (a + bL)^2 K + q + rL^2$$

Graphical Solution of the Min-Riccati and L-Riccati Equations



If $K^* > 0$ solves min-Riccati eq., then $J^*(x) = K^* x^2$, and the optimal policy satisfies

$$\mu^*(x) = \arg\min_u \left[qx^2 + ru^2 + K^*(ax + bu)^2 \right].$$

It is the linear function of x, $\mu^*(x) = L^* x$, with $L^* = -\frac{abK^*}{r + b^2 K^*}$

Computational Solution of the L-Q Problem

Value Iteration (VI) algorithms

Starting with quadratic $J_0(x) = K_0 x^2$, the VI iterates for the min-Riccati and L-Riccati eqs. are quadratic: $J_{k+1}(x) = K_{k+1}x^2 = F(K_k)x^2$, where $\{K_k\}$ is generated by

$$K_{k+1} = F(K_k) = \frac{a^2 r K_k}{r + b^2 K_k} + q, \qquad K_{k+1} = F_L(K_k) = (a + bL)^2 K_k + q + rL^2$$

Policy Iteration (PI) algorithm

Start with a linear policy $\mu^0(x) = L_0 x$. Each iteration consists of two steps: Policy evaluation and policy improvement

- Policy evaluation of $\mu^k(x) = L_k x$: Solve the L_k-Riccati eq.

$$K = F_{L_k}(K) = (a + bL_k)^2 K + q + rL_k^2$$

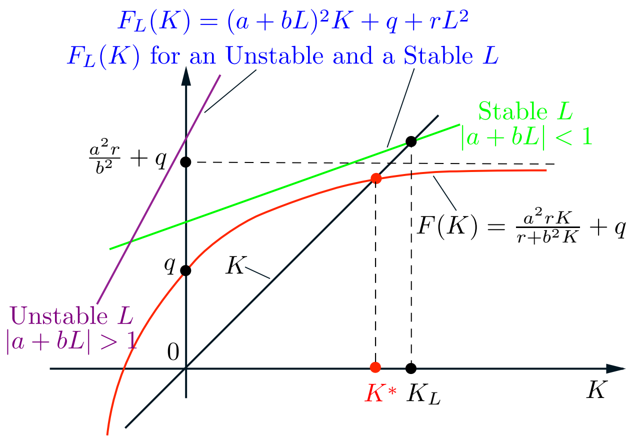
  for the solution K_{L_k}.

- Policy improvement: Obtain $\mu^{k+1}(x) = L_{k+1}x$ from

$$\mu^{k+1}(x) = \arg\min_u \{qx^2 + ru^2 + J_{\mu^k}(ax + bu)\} = \arg\min_u \{qx^2 + ru^2 + K_{L_k}(ax + bu)^2\}$$

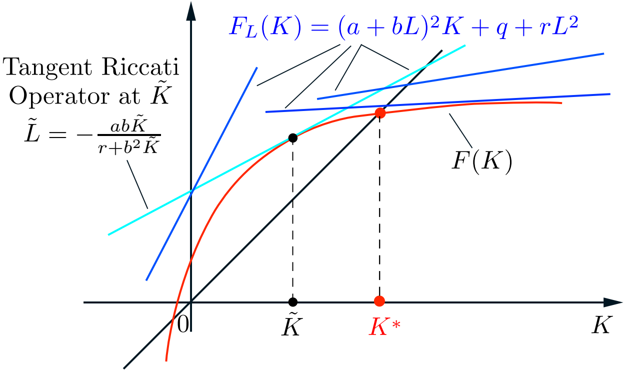
  The new policy is linear of the form $\mu^{k+1}(x) = L_{k+1}x$ with $L_{k+1} = -\frac{abK_{L_k}}{r + b^2 K_{L_k}}$

L-Riccati Equations for Stable and Unstable Policies



RICCATI EQUATIONS $K^* = F(K^*)$, $\quad K_L = F_L(K_L)$

- For $\mu(x) = Lx$ with $|a + bL| < 1$, the closed-loop linear system $x_{k+1} = (a + bL)x_k$ is stable, and we have $J_\mu(x) = K_L x^2$, where K_L solves the L-Riccati equation
- For $\mu(x) = Lx$ with $|a + bL| > 1$, the system is unstable, and we have $J_\mu(x) = \infty$ for all $x \neq 0$. (Theory and practical algorithms break down for "unstable" policies.)

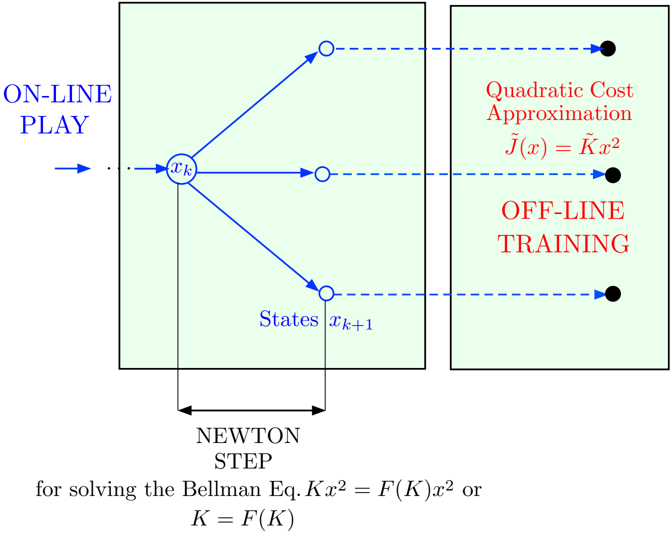


$$F(K)x^2 = \min_{u\in\Re}\left\{qx^2 + ru^2 + K(ax+bu)^2\right\}$$
$$= \min_{L\in\Re}\min_{u=Lx}\left\{qx^2 + ru^2 + K(ax+bu)^2\right\}$$
$$= \min_{L\in\Re}\left\{q + rL^2 + K(a+bL)^2\right\}x^2$$

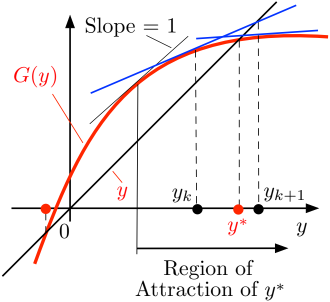
or

$$F(K) = \min_{L\in\Re} F_L(K), \quad \text{with} \quad F_L(K) = (a+bL)^2K + q + rL^2$$

At current state x_k, apply control $\tilde{\mu}(x_k) = \arg\min_u \left\{ q x_k^2 + r u^2 + \tilde{K}(a x_k + b u)^2 \right\}$



Newton's Method to Solve the Generic Fixed Point Problem $y = G(y)$



At the typical iteration k

- We linearize the problem at the current iterate y_k with a first order expansion of G,

$$G(y) \approx G(y_k) + \nabla G(y_k)(y - y_k),$$

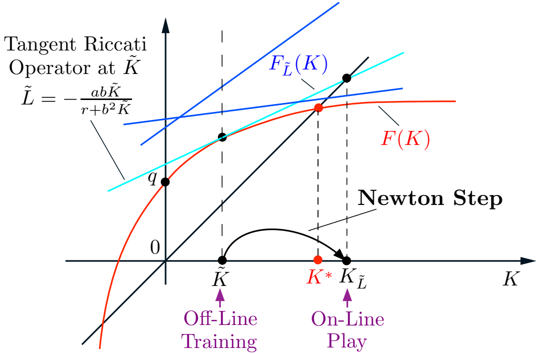
where $\nabla G(y_k)$ is the gradient of G at y_k

- We solve the linearized problem to obtain y_{k+1}:

$$y_{k+1} = G(y_k) + \nabla G(y_k)(y_{k+1} - y_k)$$

- Extends to solution of fixed point problem $y = \min\{G_1(y), \ldots, G_m(y)\}$

Visualization of Approximation in Value Space - One-Step Lookahead - No rollout



Given quadratic cost approximation $\tilde{J}(x) = \tilde{K}x^2$, we find

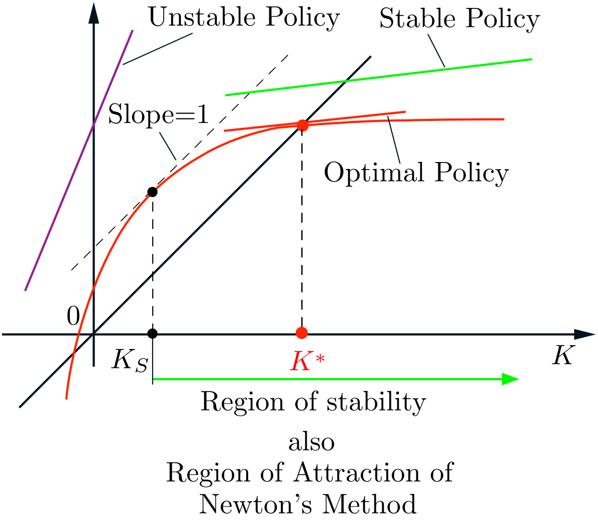
$$\tilde{L} = \arg\min_L F_L(\tilde{K})$$

to construct the one-step lookahead policy $\tilde{\mu}(x) = \tilde{L}x$

and its cost function $J_{\tilde{\mu}}(x) = K_{\tilde{L}}x^2$

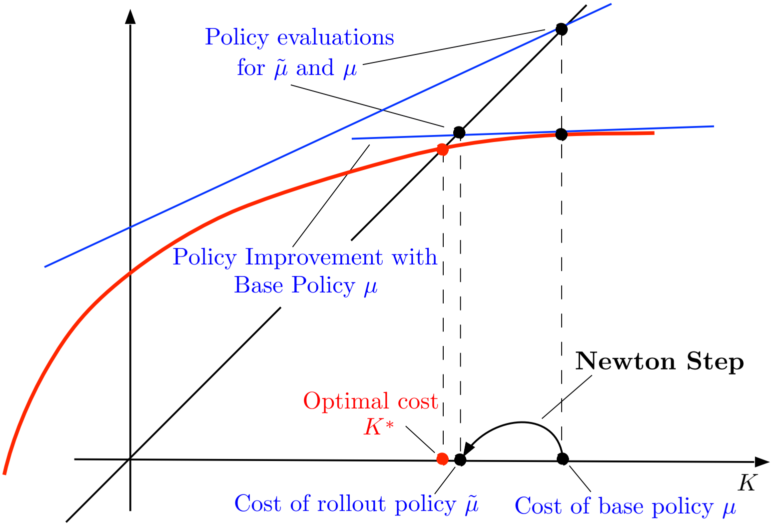
Visualization of Region of Stability of the One-Step Lookahead Policy $\tilde{\mu}$

The Set of $\tilde{K}$ for which $\tilde{\mu}$ is Stable



The start of the Newton step must be within the region of stability

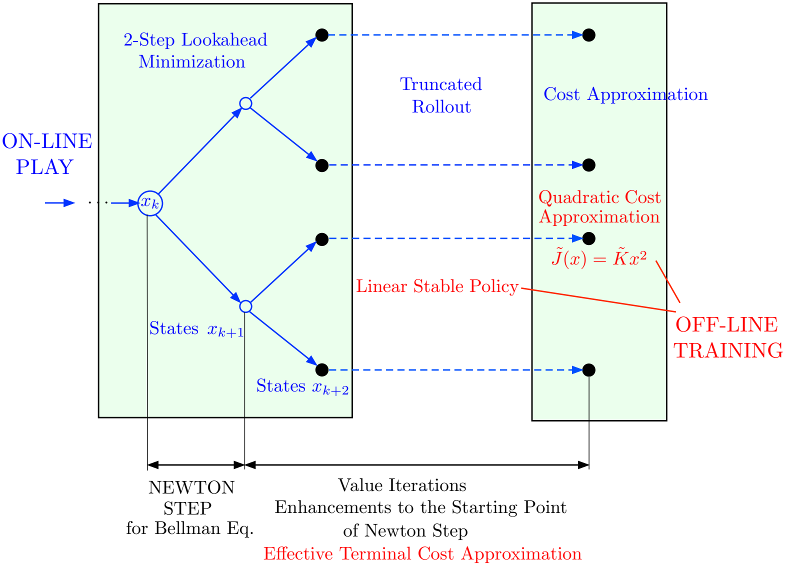
Visualization of Rollout with Stable Linear Base Policy μ: $\tilde{J} = J_\mu$



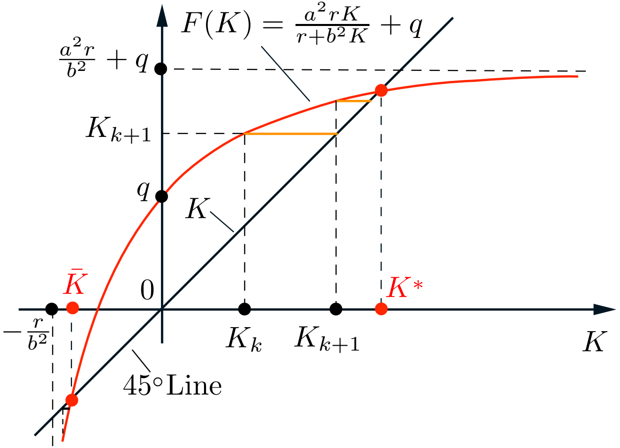
Preservation of stability: If μ is stable, $\tilde{\mu}$ is also stable

A Fifteen-Minute Break

Catch our breath and think about issues relating to the first half of the lecture.
Ask questions when you return.

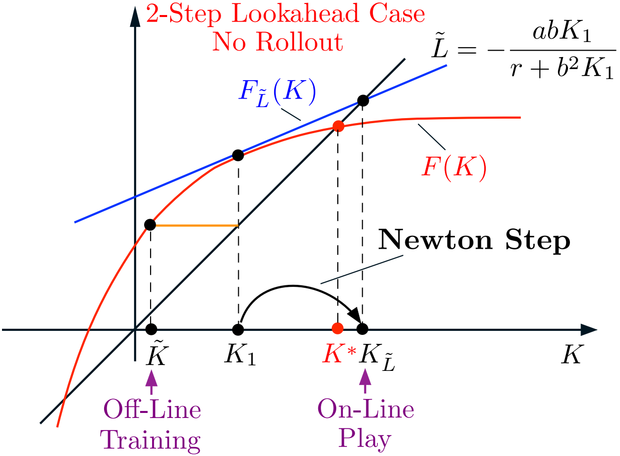


2-Step Lookahead Minimization
Truncated Rollout
Cost Approximation
ON-LINE PLAY
x_k
Quadratic Cost Approximation
$\tilde{J}(x) = \tilde{K}x^2$
Linear Stable Policy
States x_{k+1}
OFF-LINE TRAINING
States x_{k+2}
NEWTON STEP for Bellman Eq.
Value Iterations
Enhancements to the Starting Point of Newton Step
Effective Terminal Cost Approximation



Value Iteration: $K_{k+1} = F(K_k)$
from
$$J_{k+1}(x) = K_{k+1}x^2 = F(K_k)x^2$$

Visualization of Approx. in Value Space - Multi-Step Lookahead
Better Approximation and Improved Stability Properties



Multistep lookahead moves
the starting point of the Newton step closer to K^*
The longer the lookahead the better

Policy Iteration for the Linear Quadratic Problem (Repeated Rollout)

Starts with linear policy $\mu^0(x) = L_0 x$, generates sequence of linear policies $\mu^k(x) = L_k x$ with a two-step process

- Policy evaluation:

$$J_{\mu^k}(x) = K_k x^2$$

where

$$K_k = \frac{q + rL_k^2}{1 - (a + bL_k)^2}$$

- Policy improvement:

$$\mu^{k+1}(x) = L_{k+1} x$$

where

$$L_{k+1} = -\frac{abK_k}{r + b^2 K_k}$$

- Rollout is a single Newton iteration
- PI is a full-fledged Newton method for solving the Riccati equation $K = F(K)$
- An important variant, Optimistic PI, consists of repeated truncated rollout iterations
- Can be viewed as a Newton-SOR method (repeated application of a Newton step, preceded by first order VIs)

Generalization

The Newton step interpretation of approximation in value space generalizes very broadly
See the "Lessons from AlphaZero ..." textbook

- Riccati operators —> Bellman operators
- Newton's method for solving the min-Riccati equation —> Newton's method for solving the min-Bellman equation
- A mathematical point: Nondifferentiabilty of the Bellman operator is not an issue (a form of Newton's method that can deal with nondifferentiability is used; see the "Lessons from AlphaZero ..." textbook)
- Approximation in value space is a single Newton iteration, enhanced by multistep lookahead (if any), and by truncated rollout (if any)
- Rollout is a single Newton iteration starting from the cost function of the (stable) base policy
- Exact PI is a full-fledged Newton's method
- Multistep lookahead and truncated rollout enhance the stability properties of the policy produced by approximation in value space

How do we Formulate DP Problems in Practice?

An informal recipe: First define the controls, then the stages (and info available at each stage), and then the states

- Define as state x_k something that “summarizes" the past for purposes of future optimization, i.e., as long as we know x_k, all past information is irrelevant.
- Rationale: The controller applies action that depends on the state. So the state must subsume all info that is useful for decision/control.

Some examples

- In the traveling salesman problem, we need to include all the relevant info in the state (e.g., the past cities visited, and the current city). Other info, such as the costs incurred so far, need not be included in the state.
- In partial or imperfect information problems, we use “noisy" measurements for control of some quantity of interest y_k that evolves over time (e.g., the position/velocity vector of a moving object). It is correct to use I_k (the collection of all measurements up to time k) as state.
- It may also be correct to use alternative states; e.g., the conditional probability distribution $P_k(y_k \mid I_k)$. This is called belief state, and subsumes all the information that is useful for the purposes of control choice.

State Augmentation: Delays

$$x_{k+1} = f_k(x_k, x_{k-1}, u_k, u_{k-1}, w_k)$$

- Introduce additional state variables y_k and s_k, where $y_k = x_{k-1}$, $s_k = u_{k-1}$. Then

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \\ s_{k+1} \end{pmatrix} = \begin{pmatrix} f_k(x_k, y_k, u_k, s_k, w_k) \\ x_k \\ u_k \end{pmatrix}$$

- Define $\tilde{x}_k = (x_k, y_k, s_k)$ as the new state, we have

$$\tilde{x}_{k+1} = \tilde{f}_k(\tilde{x}_k, u_k, w_k)$$

- Reformulated DP algorithm: Start with $J_N^*(x_N) = g_N(x_N)$

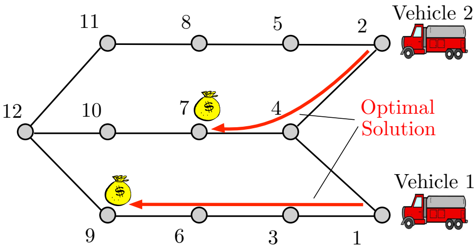
$$J_k^*(x_k, x_{k-1}, u_{k-1}) = \min_{u_k \in U_k(x_k)} E_{w_k}\Big\{g_k(x_k, u_k, w_k) + J_{k+1}^*\big(f_k(x_k, x_{k-1}, u_k, u_{k-1}, w_k), x_k, u_k\big)\Big\}$$

$$J_0^*(x_0) = \min_{u_0 \in U_0(x_0)} E_{w_0}\Big\{g_0(x_0, u_0, w_0) + J_1^*\big(f_0(x_0, u_0, w_0), x_0, u_0\big)\Big\}$$

See the textbook for other types of state augmentation (e.g., forecasts of future uncertainty)

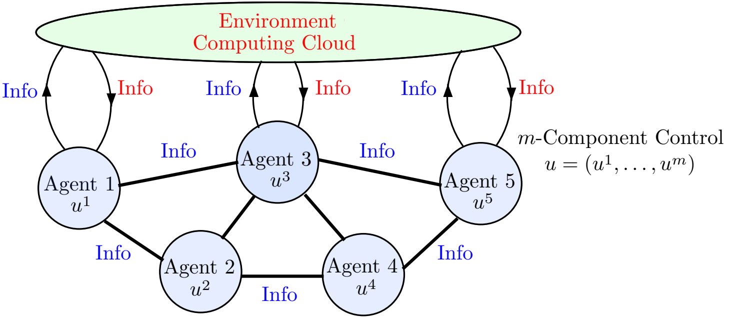
Problems with a Cost-Free and Absorbing Terminal (Goal) State

- Generally, we can view them as infinite horizon problems
- Another possibility is to convert to a finite horizon problem: Introduce as horizon an upper bound to the optimal number of stages (assuming such a bound is known)
- Add BIG penalty for not terminating before the end of the horizon



Example: Multiple vehicles move simultaneously one step at a time

- Minimize the number of moves to perform all tasks (i.e., reach the terminal state)
- How to formulate as DP? States? Controls? Terminal state? Horizon?
- Problem “size"? Astronomical, even for modest number of tasks and vehicles
- A good candidate for the multiagent framework to be introduced next

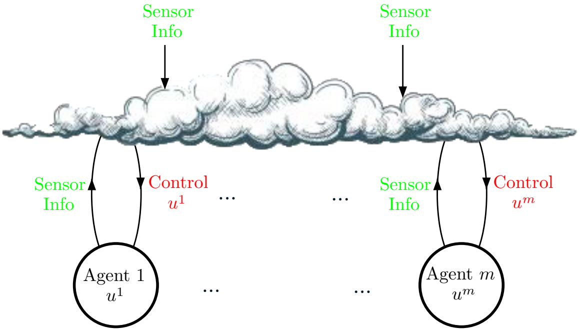


- Multiple agents collecting and sharing information selectively with each other and with an environment/computing cloud
- Agent i applies decision u^i sequentially in discrete time based on info received

The major mathematical distinction between problem structures

- The classical information pattern: Agents are fully cooperative, fully sharing and never forgetting information. Can be treated by DP
- The nonclassical information pattern: Agents are partially sharing information, and may be antagonistic. HARD because it is hard to treat by DP

Starting Point: A Classical Information Pattern (We Generalize Later)

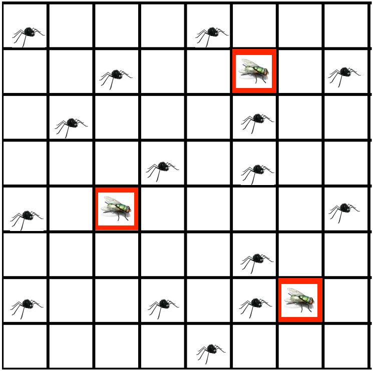


At each time: Agents have exact state info; choose their controls as function of state

Model: A discrete-time (possibly stochastic) system with state x and control u

- Decision/control has m components $u = (u^1, \ldots, u^m)$ corresponding to m "agents"
- "Agents" is just a metaphor - the important math structure is $u = (u^1, \ldots, u^m)$
- The theoretical framework is DP. We will reformulate for faster computation
  - We first aim to deal with the exponential size of the search/control space
  - Later we will discuss how to compute the agent controls in distributed fashion (in the process we will deal in part with nonclassical info pattern issues)

Spiders-and-Flies Example
(e.g., Vehicle Routing, Maintenance, Search-and-Rescue, Firefighting)



15 spiders move in 4 directions with perfect vision

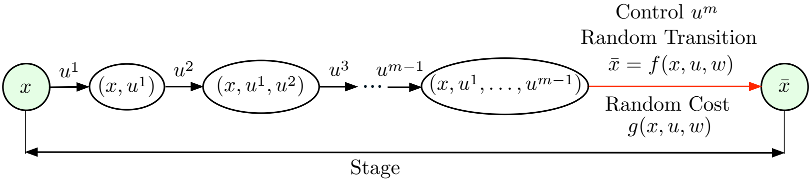
3 blind flies move randomly

Objective is to

Catch the flies in minimum time

- At each time we must select one out of $\approx 5^{15}$ joint move choices
- We will reduce to $5 \cdot 15 = 75$ (while maintaining good properties)
- Idea: Break down the control into a sequence of one-spider-at-a-time moves
- For more discussion, including illustrative videos of spiders-and-flies problems, see https://www.youtube.com/watch?v=eqbb6vVIN38&t=1654s

Reformulation Idea: Trading off Control and State Complexity (B+T NDP Book, 1996)



An equivalent reformulation - "Unfolding" the control action

- The control space is simplified at the expense of $m - 1$ additional layers of states, and corresponding $m - 1$ cost functions

$$J^1(x, u^1), J^2(x, u^1, u^2), \ldots, J^{m-1}(x, u^1, \ldots, u^{m-1})$$

- Allows far more efficient rollout (one-agent-at-a-time). This is just standard rollout for the reformulated problem (so it involves a Newton step)
- The increase in size of the state space does not adversely affect rollout (only one state and its successors are looked at each stage during on-line play)
- Complexity reduction: The one-step lookahead branching factor is reduced from n^m to $n \cdot m$, where n is the number of possible choices for each component u^i

About the Next Lecture

We will discuss special types of problem domains and reformulations, including POMDP, adaptive, and model predictive control

HOMEWORK 3 (DUE IN ONE WEEK):
EXERCISE 1.2 OF THE LATEST VERSION OF THE CLASS TEXTBOOK

READ AHEAD SECTION 1.6 OF THE LATEST VERSION OF THE CLASS TEXTBOOK

This is a good time to watch the summary videolecture at
https://www.youtube.com/watch?v=A7OGgpuRnuo (1-hour version)
of the book
Lessons for AlphaZero for Optimal, Model Predictive, and Adaptive Control
Also the multiagent videolecture at
https://www.youtube.com/watch?v=eqbb6vVIN38