Topics in Reinforcement Learning: AlphaZero, ChatGPT, Neuro-Dynamic Programming, Model Predictive Control, Discrete Optimization
Arizona State University
Course CSE 691, Spring 2024

Links to Class Notes, Videolectures, and Slides at
http://web.mit.edu/dimitrib/www/RLbook.html

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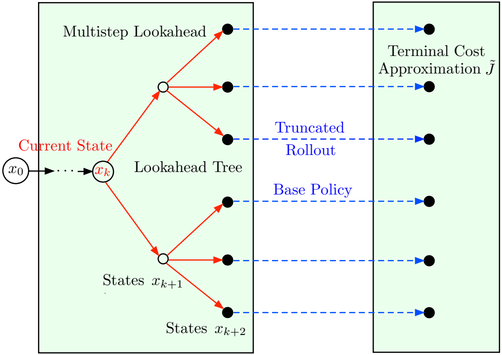
Lecture 6
Deterministic Problems: Multistep Approximation in Value Space, Constrained Rollout, Multiagent Rollout

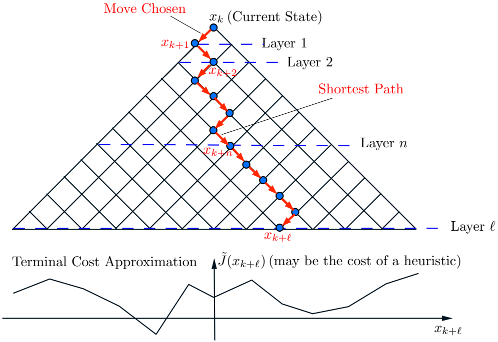
Outline

Multistep Approximation in Value Space - The General Case



- Special case: No rollout. The general multistep approximation in value space scheme.
- Special case: Pure multistep rollout. No terminal cost and no truncation.
- WE TAKE IT AS FACT: Longer lookahead improves performance (but is costly).
- OUR STRATEGY: Extend the lookahead as much as the comp. budget allows.
- One idea: Truncated rollout (a cheap extension of the lookahead length).
- Another computation-saving idea: Selectively prune the lookahead tree.

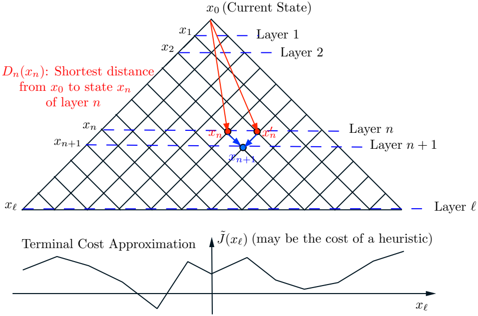
Multistep Lookahead in Deteministic Problems



We obtain a trajectory $\{x_k, x_{k+1}, \dots, x_{k+\ell}\}$ that minimizes the shortest distance from x_k to $x_{k+\ell}$ PLUS $\tilde{J}(x_{k+\ell})$. We then use the first move $x_k \to x_{k+1}$.

- All the shortest path problems from x_k to $x_{k+\ell}$ can be solved simultaneously by backward DP (start from layer ℓ go towards x_k).
- An important alternative is the forward DP algorithm.
- It is the same as the backwards DP algorithm with the direction of the arcs reversed (start from x_k go towards layer ℓ - see the next slide).

Forward DP Algorithm and Iterative Deepening [$\tilde{J}(x)$ is given for all x]

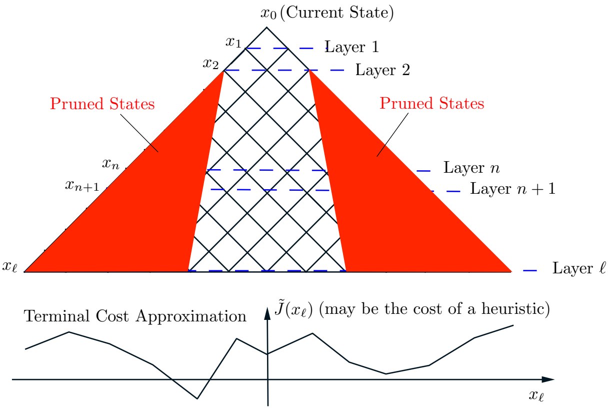


- The "forward" DP algorithm: The shortest distances $D_{n+1}(x_{n+1})$ to layer $n+1$ states are obtained from the shortest distances $D_n(x_n)$ to layer n states as follows:

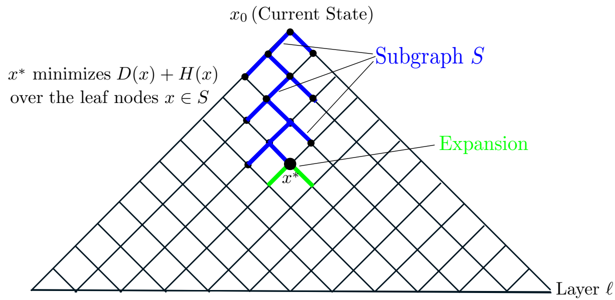
$$D_{n+1}(x_{n+1}) = \min_{x_n} \Big[(\text{Cost } x_n \to x_{n+1}) + D_n(x_n) \Big]$$

- Solution of the ℓ-step lookahead problem: The shortest path to the state x_ℓ^* of layer ℓ that minimizes $D_\ell(x_\ell) + \tilde{J}(x_\ell)$.
- Iterative deepening: Solve the n-step lookahead problem before solving the $(n+1)$-step lookahead problem.
- This is an "anytime" algorithm (returns a feasible solution even if it is interrupted).

Iterative Deepening with Tree Pruning



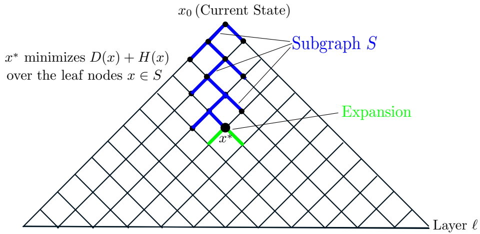
- Iterative deepening can be "enhanced" by pruning states $\hat{x}_n$ such that the n-step lookahead cost $D_n(\hat{x}_n) + \tilde{J}(\hat{x}_n)$ is "far from the minimum" over x_n.
- We prune as we go: Prune states in layer n before pruning states in layer $n+1$.
- Runs the risk of overpruning: Some pruned states may be "good" in hindsight.
- Should we go back and check for overpruning? How?



We use a less regular graph, which is expanded at each iteration based on a shortest path computation

- At the start of an iteration, we have an acyclic connected subgraph S rooted at x_0.
- We compute the shortest distance $D(x)$ from x_0 to all $x \in S$, going through S.
- We find a leaf node $x^* \in S$ that minimizes $D(x) + H(x)$, where $H(x)$ is a “heuristic distance" from x to layer ℓ.
- Expand x^* to enlarge S and start the next iteration (or stop if x^* is in layer ℓ).

Incremental Multistep Rollout - Some Details

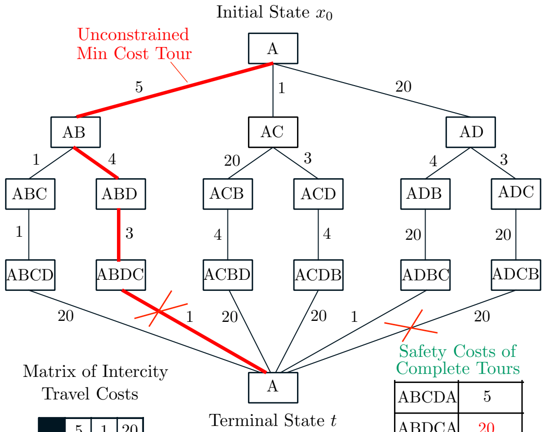


- At the start of an iteration, we have an acyclic connected subgraph S rooted at x_0.
- We minimize $D(x) + H(x)$ over all leaf nodes $x \in S$.
- We expand the minimizing node x^* to form the new subgraph.
- The computation of the shortest distances $D(x)$ is done progressively with the forward DP algorithm as the subgraph S expands.
- Example of $H(x)$: The cost of a base heuristic that starts from x and ends at some node x_ℓ of layer ℓ, plus $\tilde{J}(x_\ell)$, plus an extra term that favors paths with few hops that encourages backtracking e.g., $\delta \cdot$ (number of hops from x_0 to x), where $\delta > 0$.
- For $\delta = 0$, we get max pruning: S ends up being “long and skinny". For $\delta \approx \infty$, we get min pruning: S ends up being as “fat" as possible.

Constrained Rollout - Main Ideas

Applies to problems with additional constraints on the entire optimal trajectory

- Greatly expands the range of applications of rollout
- For example it applies to intractable discrete optimization problems (e.g., shortest path problems with a limit on the number of hops).
- It is similar to unconstrained rollout: As we expand the rollout path, we exclude from consideration the Q-factors that correspond to constraint violation.
- Guarantees cost improvement over the base heuristic under appropriate conditions (modified versions of sequential consistency, sequential improvement, or use of a fortified version).



Matrix of Intercity Travel Costs

	5	1	20
20		1	4
1	20		1
20	4	3	

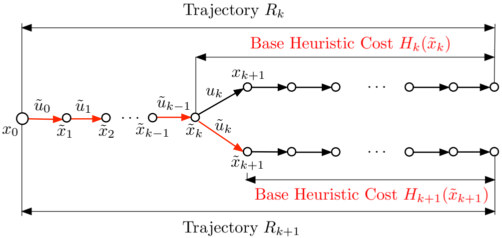
Constraint: Tour Safety ≤ 10

Safety Costs of Complete Tours

ABCDA	5
ABDCA	20
ACBDA	4
ACDBA	3
ADBCA	1
ADCBA	15

Find a minimum cost tour subject to a safety constraint

Deterministic Rollout with Trajectory Constraint: Basic Idea



Review of the unconstrained rollout algorithm:

- Construct sequence of trajectories $\{T_0, T_1, \ldots, T_N\}$ with monotonically nonincreasing cost (assuming a sequential improvement condition).
- For each k, the trajectories $T_k, T_{k+1}, \ldots, T_N$ share the same initial portion $(x_0, \tilde{u}_0, \ldots, \tilde{u}_{k-1}, \tilde{x}_k)$.
- The base heuristic is used to generate candidate trajectories that correspond to the controls $u_k \in U_k(x_k)$.
- The next trajectory T_{k+1} is the candidate trajectory that has min cost.

To deal with a trajectory constraint $T \in C$, we discard all the candidate trajectories that violate the constraint, and we choose T_{k+1} to be the best of the remaining trajectories.

Deterministic Problems with Constraints: Definition

- Consider a deterministic optimal control problem with system $x_{k+1} = f_k(x_k, u_k)$.
- A complete trajectory is a sequence

$$T = (x_0, u_0, x_1, u_1, \dots, u_{N-1}, x_N)$$

- Problem:

$$\min_{T \in C} G(T)$$

where G is a given cost function and C is a given constraint set of trajectories.

State augmentation idea for rollout

- Redefine the state to be the partial trajectory

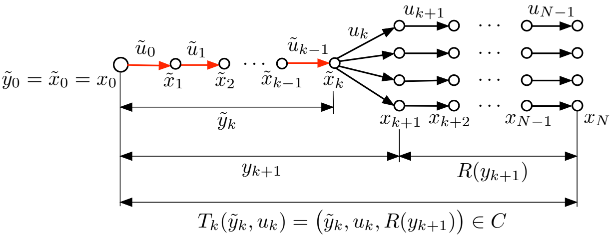
$$y_k = (x_0, u_0, x_1, \dots, u_{k-1}, x_k)$$

- Partial trajectory evolves according to a redefined system equation:

$$y_{k+1} = \big(y_k, u_k, f_k(x_k, u_k)\big)$$

- The problem becomes to minimize $G(y_N)$ subject to the constraint $y_N \in C$.

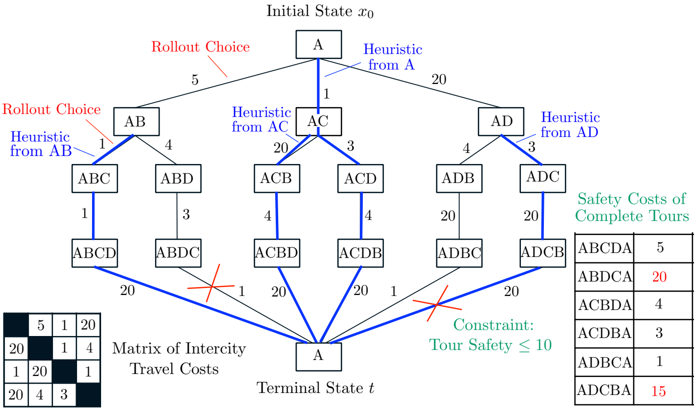
Rollout Algorithm - Partial Trajectory-Dependent Base Heuristic



- Given $\tilde{y}_k = \{\tilde{x}_0, \tilde{u}_0, \tilde{x}_1, \tilde{u}_1, \ldots, \tilde{u}_{k-1}, \tilde{x}_k\}$ consider all controls u_k and corresponding next states x_{k+1}.
- Extend $\tilde{y}_k$ to obtain the partial trajectories $y_{k+1} = \big(\tilde{y}_k, u_k, x_{k+1}\big)$, for $u_k \in U_k(x_k)$.
- Run the base heuristic from each y_{k+1} to obtain the partial trajectory $R(y_{k+1})$.
- Join the partial trajectories y_{k+1} and $R(y_{k+1})$ to obtain complete trajectories denoted by $T_k(\tilde{y}_k, u_k) = \big(\tilde{y}_k, u_k, R(y_{k+1})\big)$
- Find the set of controls $\tilde{U}_k(\tilde{y}_k)$ for which $T_k(\tilde{y}_k, u_k)$ is feasible, i.e., $T_k(\tilde{y}_k, u_k) \in C$
- Choose the control $\tilde{u}_k \in \tilde{U}_k(\tilde{y}_k)$ according to the minimization

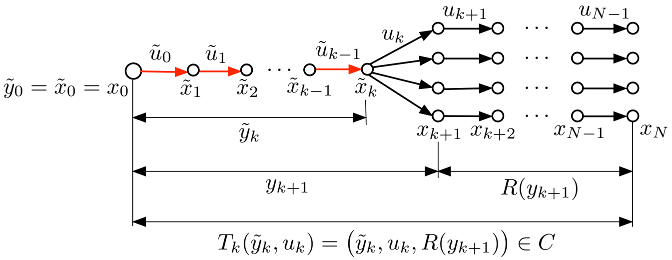
$$\tilde{u}_k \in \arg \min_{u_k \in \tilde{U}_k(\tilde{y}_k)} G\big(T_k(\tilde{y}_k, u_k)\big)$$

Constrained Traveling Salesman Example

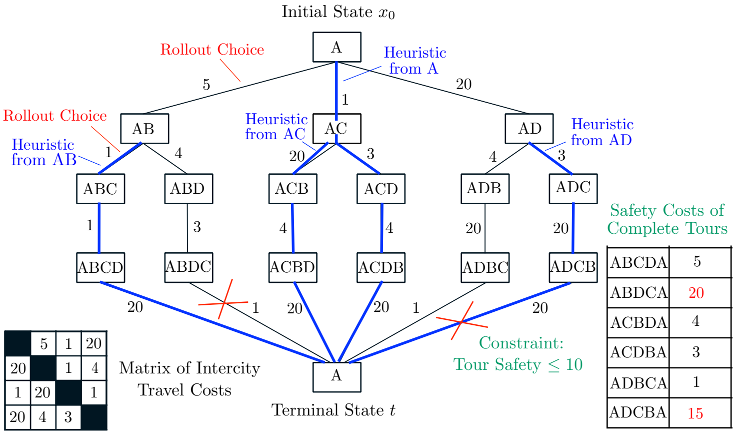


- Rollout at A: Considers partial tours AB, AC, and AD; Obtains the complete tours ABCDA, ACBDA, and ADCBA; Discards ADCBA as being infeasible; Compares ABCDA and ACBDA, finds ABCDA to have smaller cost, and selects AB.
- Rollout at AB: Considers the partial tours ABC and ABD; Obtains the complete tours ABCDA and ABDCA; Discards ABDCA as being infeasible; Selects the complete tour ABCDA.

Constrained Rollout Algorithm Properties



- The notions of sequential consistency and sequential improvement apply. Their definition includes that the set of “feasible" controls $\tilde{U}_k(\tilde{y}_k)$ is nonempty for all k.
- Sequential improvement condition: The min heuristic Q-factor over $\tilde{U}_k(\tilde{y}_k)$ is no larger than the heuristic cost at $\tilde{y}_k$ (see the “Course in RL" textbook).
- Fortified version (if sequential improvement does not hold; see the notes):
  - Maintains the “tentative best" trajectory, and follows it up to generating a better trajectory through rollout.
  - Has the cost improvement property, assuming the base heuristic generates a feasible trajectory starting from the initial condition $\tilde{y}_0 = x_0$.
- Multiagent version: Selects one-control-component-at-a-time (apply constrained rollout to the equivalent reformulation, i.e., the one with control space “unfolded").



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- The heuristic is not sequentially consistent at A, but it is sequentially improving.
- If we change the D→A cost to 25, the heuristic is not sequentially improving at A, and the cost improvement property is lost.
- If we change the D→A cost to 25 and we add fortification, the rollout algorithm at A sticks with the initial tentative best trajectory ACDBA, and rejects ABCDA.

A Retrospective Summary on Deterministic Constrained Rollout

Structural components

(1) Trajectories T consisting of a sequence of decisions defined by a layered/optimal control graph

(2) A cost function $G(T)$ to rank trajectories

(3) A constraint $T \in C$ to determine feasibility of trajectories

(4) A base heuristic that starts from a partial trajectory and generates a complete trajectory

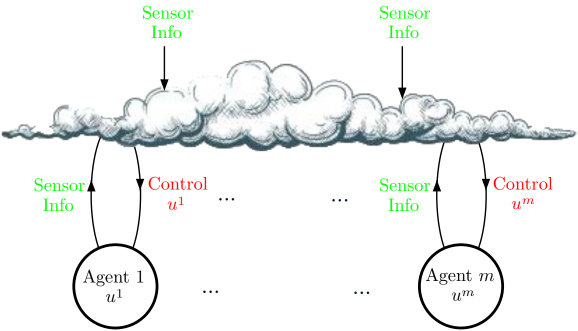
Given (1)

The choices of (2), (3), and (4) are independent of each other

In particular, given (1)-(3):

We can try several different base heuristics or a superheuristic

Multiagent Problems: Review



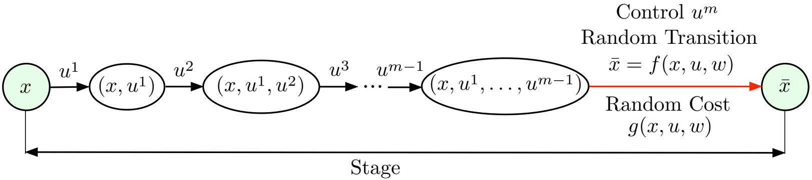
Classical information pattern

At each time: Agents have exact state info; choose their controls as function of state

Model: A discrete-time (possibly stochastic) system with state x and control u

- Decision/control has m components $u = (u^1, \ldots, u^m)$ corresponding to m "agents"
- "Agents" is just a metaphor - the important math structure is $u = (u^1, \ldots, u^m)$
- We will reformulate the problem so that rollout can be done much faster

Reformulation Idea: Trading off Control and State Complexity (B+T NDP Book, 1996)



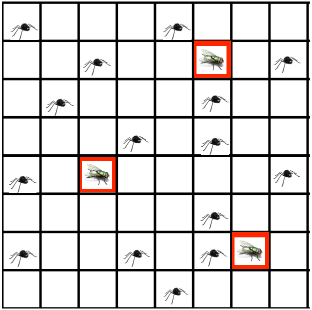
An equivalent reformulation - "Unfolding" the control action

- The control space is simplified at the expense of $m-1$ additional layers of states, and corresponding $m-1$ cost functions

$$J^1(x, u^1), J^2(x, u^1, u^2), \ldots, J^{m-1}(x, u^1, \ldots, u^{m-1})$$

- Allows far more efficient rollout (one-agent-at-a-time). This is just standard rollout for the reformulated problem (so it involves a Newton step)
- The increase in size of the state space does not adversely affect rollout (only one state and its successors are looked at each stage during on-line play)
- Complexity reduction: The one-step lookahead branching factor is reduced from n^m to $n \cdot m$, where n is the number of possible choices for each component u^i

Spiders-and-Flies Example (e.g., Vehicle Routing, Maintenance, Search-and-Rescue, Firefighting)



15 spiders move in 4 directions with perfect vision

3 blind flies move randomly

Objective is to

Catch the flies in minimum time

- In the original problem, at each time we must consider $\approx 5^{15}$ joint moves
- In the reformulated problem, we break down the control into a sequence of one-spider-at-a-time moves
- Thus, we need to consider only $5 \cdot 15 = 75$ (while maintaining the rollout cost improvement property)
- For more discussion, including illustrative videos of spiders-and-flies problems, see https://www.youtube.com/watch?v=eqbb6vVIN38&t=1654s Also Section 2.6 of the course textbook

Final Notes

The material of today's lecture is covered in the "Lessons from AlphaZero ..." monograph as well as the “Course in RL" textbook

In the next lecture we will cover:

- Stochastic Rollout.
- Monte Carlo Tree Search.
- Rollout for infinite spaces problems.