Massachusetts Institute of Technology
Department of Physics

Physics 8.962 Spring 2002

Tensor Calculus, Part 2

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1 Introduction

The first set of 8.962 notes, *Introduction to Tensor Calculus for General Relativity*, discussed tensors, gradients, and elementary integration. The current notes continue the discussion of tensor calculus with orthonormal bases and commutators (§2), parallel transport and geodesics (§3), and the Riemann curvature tensor (§4).

2 Orthonormal Bases, Tetrads, and Commutators

A vector basis is said to be orthonormal at point **X** if the dot product is given by the Minkowski metric at that point:

$$\{\vec{e}_{\hat{\mu}}\} \text{ is orthonormal if and only if } \vec{e}_{\hat{\mu}} \cdot \vec{e}_{\hat{\nu}} = \eta_{\mu\nu} \ . \tag{1}$$

(We have suppressed the implied subscript **X** for clarity.) Note that we will always place a hat over the index for any component of an orthonormal basis vector. The smoothness properties of a manifold imply that it is always possible to choose an orthonormal basis at any point in a manifold. One simply choose a basis that diagonalizes the metric g and furthermore reduces it to the normalized Minkowski form. Indeed, there are infinitely many orthonormal bases at **X** related to each other by Lorentz transformations. Orthonormal bases correspond to locally inertial frames.

For each basis of orthonormal vectors there is a corresponding basis of orthonormal one-forms related to the basis vectors by the usual duality condition:

$$\left\langle \tilde{e}^{\hat{\mu}}, \vec{e}_{\hat{\nu}} \right\rangle = \delta^{\mu}{}_{\nu} \ . \tag{2}$$

The existence of orthonormal bases at one point is very useful in providing a locally inertial frame in which to present the components of tensors measured by an observer at rest in that frame. Consider an observer with 4-velocity $\vec{V}$ at point $\mathbf{X}$. Since $\vec{V}\cdot\vec{V} = -1$, the observer's rest frame has timelike orthonormal basis vector $\vec{e}_{\hat{0}} = \vec{V}$. The observer has a set of orthonormal space axes given by a set of spatial unit vectors $\vec{e}_{\hat{i}}$. For a given $\vec{e}_{\hat{0}}$, there are of course many possible choices for the spatial axes that are related by spatial rotations. Each choice of spatial axes, when combined with the observer's 4-velocity, gives an orthonormal basis or tetrad. Thus, an observer carries along an orthonormal bases that we call the **observer's tetrad**. This basis is the natural one for splitting vectors, one-forms, and tensors into timelike and spacelike parts. We use the observer's tetrad to extract physical, measurable quantities from geometric, coordinate-free objects in general relativity.

For example, consider a particle with 4-momentum $\vec{P}$. The energy in the observer's instantaneous inertial local rest frame is $E = -\vec{V}\cdot\vec{P} = -\vec{e}_{\hat{0}}\cdot\vec{P} = \langle\tilde{e}^{\hat{0}},\vec{P}\rangle$. The observer can define a $(2,0)$ projection tensor

$$\mathsf{h} \equiv \mathsf{g}^{-1} + \vec{V}\otimes\vec{V} \tag{3}$$

with components (in any basis) $h^{\alpha\beta} = g^{\alpha\beta} + V^\alpha V^\beta$. This projection tensor is essentially the inverse metric on spatial hypersurfaces orthogonal to $\vec{V}$; the corresponding $(0,2)$ tensor is $h_{\mu\nu} = g_{\alpha\mu}g_{\beta\nu}h^{\alpha\beta}$. The reader can easily verify that $h_{\mu\nu}V^\mu = h_{\mu\nu}V^\nu = 0$, hence in the observer's tetrad, $h^{\hat{\mu}\hat{\nu}} = h_{\hat{\mu}\hat{\nu}} = \mathbf{diag}(0,1,1,1)$. Then, the spatial momentum components follow from $P^{\hat{i}} = \langle\tilde{e}^{\hat{i}},\vec{P}\rangle = P_{\hat{i}} = \vec{e}_{\hat{i}}\cdot\vec{P}$. (Normally it is meaningless to equate components of one-forms and vectors since they cannot be equal in all bases. Here we are restricting ourselves to a single basis — the observer's tetrad — where it happens that spatial components of one-forms and vectors are equal.) Note that $P^{\hat{i}}\vec{e}_{\hat{i}} = \mathsf{h}(\mathsf{g}(\vec{P}\,))$: the spatial part of the momentum is extracted using h. Thus, in any basis, $P^\mu = EV^\mu + h^\mu{}_\nu P^\nu$ splits $\vec{P}$ into parts parallel and perpendicular to $\vec{V}$. (Note $h^\mu{}_\nu \equiv g_{\kappa\nu}h^{\mu\kappa}$.)

2.1 Tetrads

If one can define an orthonormal basis for the tangent space at any point in a manifold, then one can define a set of orthonormal bases for **every** point in the manifold. In this way, equation (1) applies everywhere. At all spacetime points, the dot product has been reduced to the Minkowski form: $g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$. One then has an orthonormal basis, or tetrad, for all points of spacetime.

If spacetime is not flat, how can we reduce the metric at every point to the Minkowski form? Doesn't that require a globally flat, Minkowski spacetime? How can one have the Minkowski metric without having Minkowski spacetime?

The resolution of this paradox lies in the fact that the metric we introduced in a coordinate basis has at least three different roles, and only one of them is played by $\eta_{\hat{\mu}\hat{\nu}}$. First, the metric gives the dot product: $\vec{A}\cdot\vec{B} = g_{\mu\nu}A^\mu B^\nu = \eta_{\hat{\mu}\hat{\nu}}A^{\hat{\mu}}B^{\hat{\nu}}$. Both $g_{\mu\nu}$

and $\eta_{\hat{\mu}\hat{\nu}}$ fulfill this role. Second, the metric components in a coordinate basis give the connection through the well-known Christoffel formula involving the partial derivatives of the metric components. Obviously since $\eta_{\hat{\mu}\hat{\nu}}$ has zero derivatives, it cannot give the connection. Third, the metric in a coordinate basis gives spacetime length and time through $d\vec{x} = dx^\mu \vec{e}_\mu$. Combining this with the dot product gives the line element, $ds^2 = d\vec{x} \cdot d\vec{x} = g_{\mu\nu} dx^\mu dx^\nu$. This formula is true only in a coordinate basis!

Usually when we speak of "metric" we mean the metric in a coordinate basis, which relates coordinate differentials to the line element: $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$. An orthonormal basis, unless it is also a coordinate basis, does not have enough information to provide the line element (or the connection). To determine these, we must find a linear transformation from the orthonormal basis to a coordinate basis:

$$\vec{e}_\mu = E^{\hat{\mu}}{}_\mu \vec{e}_{\hat{\mu}} \ . \tag{4}$$

The coefficients $E^{\hat{\mu}}{}_\mu$ are called the **tetrad components**. Note that $\hat{\mu}$ labels the (tetrad) basis vector while μ labels the component in some coordinate system (which may have no relation at all to the orthonormal basis). For a given orthonormal basis, $E^{\hat{\mu}}{}_\mu$ may be regarded as (the components of) a set of 4 one-form fields, one one-form $\tilde{E}^{\hat{\mu}} = E^{\hat{\mu}}{}_\mu \tilde{e}^\mu$ for each value of $\hat{\mu}$. Note that the tetrad components are *not* the components of a (1,1) tensor because of the mixture of two different bases.

The tetrad may be inverted in the obvious way:

$$\vec{e}_{\hat{\mu}} = E^\mu{}_{\hat{\mu}} \vec{e}_\mu \quad \text{where} \quad E^\mu{}_{\hat{\mu}} E^{\hat{\mu}}{}_\nu = \delta^\mu{}_\nu \ . \tag{5}$$

The dual basis one-forms are related by the tetrad and its inverse as for any change of basis: $\tilde{e}^\mu = E^\mu{}_{\hat{\mu}} \tilde{e}^{\hat{\mu}}$, $\tilde{e}^{\hat{\mu}} = E^{\hat{\mu}}{}_\mu \tilde{e}^\mu$,

The metric components in the coordinate basis follow from the tetrad components:

$$g_{\mu\nu} = \vec{e}_\mu \cdot \vec{e}_\nu = \eta_{\hat{\mu}\hat{\nu}} E^{\hat{\mu}}{}_\mu E^{\hat{\nu}}{}_\nu \tag{6}$$

or $g = E^T \eta E$ in matrix notation. Sometimes the tetrad is called the "square root of the metric." Equation (6) is the key result allowing us to use orthonormal bases in curved spacetime.

To discuss the curvature of a manifold we first need a connection relating nearby points in the manifold. If there exists any basis (orthonormal or not) such that $\langle \tilde{e}^\lambda, \widetilde{\nabla} \vec{e}_\mu \rangle \equiv \Gamma^\lambda{}_{\mu\nu} \tilde{e}^\nu = 0$ everywhere, then the manifold is indeed flat. However, the converse is not true: if the basis vectors rotate from one point to another even in a flat space (e.g. the polar coordinate basis in the plane) the connection will not vanish. Thus we will need to compute the connection and later look for additional quantities that give an invariant (basis-free) meaning to curvature. First we examine a more primitive object related to the gradient of vector fields, the commutator.

2.2 Commutators

The difference between an orthonormal basis and a coordinate basis arises immediately when one considers the commutator of two vector fields, which is a vector that may symbolically be defined by

$$[\vec{A},\vec{B}\,] \equiv \nabla_A \nabla_B - \nabla_B \nabla_A \tag{7}$$

where ∇_A is the directional derivative ($\nabla_A = A^\mu \partial_\mu$ in a coordinate basis). Equation (7) introduces a new notation and new concept of a vector since the right-hand side consists solely of differential operators with no arrows! To interpret this, we rewrite the right-hand side in a coordinate basis using, e.g., $\nabla_A \nabla_B f = A^\mu \partial_\mu (B^\nu \partial_\nu f)$ (where f is any twice-differentiable scalar field):

$$[\vec{A},\vec{B}\,] = \left(A^\mu \frac{\partial B^\nu}{\partial x^\mu} - B^\mu \frac{\partial A^\nu}{\partial x^\mu} \right) \frac{\partial}{\partial x^\nu} \ . \tag{8}$$

This is equivalent to a vector because $\{\partial/\partial x^\nu\}$ provide a coordinate basis for vectors in the formulation of differential geometry introduced by Cartan. Given our heuristic approach to vectors as objects with magnitude and direction, it seems strange to treat a partial derivative as a vector. However, Cartan showed that directional derivatives form a vector space isomorphic to the tangent space of a manifold. Following him, differential geometry experts replace our coordinate basis vectors $\vec{e}_\mu$ by $\partial/\partial x^\mu$. (MTW introduce this approach in Chapter 8. On p. 203, they write $\vec{e}_\alpha = \partial \mathcal{P}/\partial x^\alpha$ where $\mathcal{P}$ refers to a point in the manifold, as a way to indicate the association of the tangent vector and directional derivative.) With this choice, vectors become differential operators (e.g. $\vec{A} = A^\mu \partial_\mu$) and thus the commutator of two vector fields involves derivatives. However, we need not follow the Cartan notation. It is enough for us to define the commutator of two vectors by its components in a coordinate basis,

$$[\vec{A},\vec{B}\,] = (A^\mu \partial_\mu B^\nu - B^\mu \partial_\mu A^\nu)\vec{e}_\nu \quad \text{in a coordinate basis,} \tag{9}$$

where the partial derivative operators act only on B^ν and A^ν but not on $\vec{e}_\nu$.

Equation (9) implies

$$[\vec{A},\vec{B}\,] = \nabla_A \vec{B} - \nabla_B \vec{A} + T^\mu{}_{\alpha\beta} A^\alpha B^\beta \vec{e}_\mu \ , \tag{10}$$

where $T^\mu{}_{\alpha\beta} \equiv \Gamma^\mu{}_{\alpha\beta} - \Gamma^\mu{}_{\beta\alpha}$ in a coordinate basis is a quantity called the torsion tensor. The reader may easily show that the torsion tensor also follows from the commutator of covariant derivatives applied to any twice-differentiable scalar field,

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) f = T^\mu{}_{\alpha\beta} \nabla_\mu f \tag{11}$$

This equation shows that the torsion is a tensor even though the connection is not. The torsion vanishes by assumption in general relativity. This is a statement of physics, not mathematics. Other gravity theories allow for torsion to incorporate possible new physical effects beyond Einstein gravity.

The basis vector fields $\vec{e}_\mu(x)$ are vector fields, so let us examine their commutators. From equation (9) or (10), in an coordinate basis, the commutators vanish identically (even if the torsion does not vanish):

$$[\vec{e}_\mu, \vec{e}_\nu] = 0 \quad \text{in a coordinate basis} . \tag{12}$$

The vanishing of the commutators occurs because the coordinate basis vectors are dual to an integrable basis of one-forms: $\tilde{e}^\mu = \widetilde{\nabla} x^\mu$ for a set of 4 scalar fields x^μ. It may be shown that this integrability condition (i.e. that the basis one-forms may be integrated to give functions) is equivalent to equation (12) (see Wald 1984, problem 5 of Chapter 2).

Now let us examine the commutator for an orthonormal basis. We use equation (9) by expressing the tetrad components in a coordinate basis using equation (5). The result is

$$[\vec{e}_{\hat\mu}, \vec{e}_{\hat\nu}] = \partial_{\hat\mu}\vec{e}_{\hat\nu} - \partial_{\hat\nu}\vec{e}_{\hat\mu} \equiv \omega^{\hat\alpha}{}_{\hat\mu\hat\nu}\vec{e}_{\hat\alpha} \ , \tag{13}$$

where $\partial_{\hat\mu} \equiv E^\mu{}_{\hat\mu}\partial_\mu$. Equation (13) defines the **commutator basis coefficients** $\omega^{\hat\alpha}{}_{\hat\mu\hat\nu}$ (cf. MTW eq. 8.14). Using equations (5), (12), and (13), one may show

$$\omega^{\hat\alpha}{}_{\hat\mu\hat\nu} = E^{\hat\alpha}{}_\alpha \left(\nabla_{\hat\mu} E^\alpha{}_{\hat\nu} - \nabla_{\hat\nu} E^\alpha{}_{\hat\mu} \right) = E^\mu{}_{\hat\mu} E^\nu{}_{\hat\nu} \left(\partial_\mu E^{\hat\alpha}{}_\nu - \partial_\nu E^{\hat\alpha}{}_\mu \right) \ . \tag{14}$$

In general the commutator basis coefficients do not vanish. Despite the appearance of a second (coordinate) basis, the commutator basis coefficients are independent of any other basis besides the orthonormal one. The coordinate basis is introduced solely for the convenience of partial differentiation with respect to the coordinates.

The commutator basis coefficients carry information about how the tetrad rotates as one moves to nearby points in the manifold. It is useful practice to derive them for the orthonormal basis $\{\vec{e}_{\hat r}, \vec{e}_{\hat\theta}\}$ in the Euclidean plane.

2.3 Connection for an orthonormal basis

The connection for the basis $\{\vec{e}_{\hat\mu}\}$ is defined by

$$\partial_{\hat\nu}\vec{e}_{\hat\mu} \equiv \Gamma^{\hat\alpha}{}_{\hat\mu\hat\nu}\vec{e}_{\hat\alpha} \ . \tag{15}$$

(The placement of the lower subscripts on the connection agrees with MTW but is reversed compared with Wald and Carroll.) From the local flatness theorem (metric compatibility with covariant derivative) discussed in the first set of notes,

$$\nabla_{\hat\alpha} g_{\hat\mu\hat\nu} = E^\alpha{}_{\hat\alpha}\partial_\alpha g_{\hat\mu\hat\nu} - \Gamma^{\hat\beta}{}_{\hat\mu\hat\alpha} g_{\hat\beta\hat\nu} - \Gamma^{\hat\beta}{}_{\hat\nu\hat\alpha} g_{\hat\mu\hat\beta} = 0 \ . \tag{16}$$

In an orthonormal basis, $g_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\mu}\hat{\nu}}$ is constant so its derivatives vanish. We conclude that, in an orthonormal basis, the connection is antisymmetric on its first two indices:

$$\Gamma_{\hat{\mu}\hat{\nu}\hat{\alpha}} = -\Gamma_{\hat{\nu}\hat{\mu}\hat{\alpha}} \ , \quad \Gamma_{\hat{\mu}\hat{\nu}\hat{\alpha}} \equiv g_{\hat{\mu}\hat{\beta}}\Gamma^{\hat{\beta}}{}_{\hat{\nu}\hat{\alpha}} = \eta_{\hat{\mu}\hat{\beta}}\Gamma^{\hat{\beta}}{}_{\hat{\nu}\hat{\alpha}} \ . \tag{17}$$

In an orthonormal basis, the connection is *not*, in general, symmetric on its last two indices. (That is true only in a coordinate basis.)

Another equation for the connection coefficients comes from combining equations (13) with equation (15):

$$\omega_{\hat{\alpha}\hat{\mu}\hat{\nu}} = -\Gamma_{\hat{\alpha}\hat{\mu}\hat{\nu}} + \Gamma_{\hat{\alpha}\hat{\nu}\hat{\mu}} \ , \quad \omega_{\hat{\alpha}\hat{\mu}\hat{\nu}} \equiv g_{\hat{\alpha}\hat{\beta}}\omega^{\hat{\beta}}{}_{\hat{\mu}\hat{\nu}} = \eta_{\hat{\alpha}\hat{\beta}}\omega^{\hat{\beta}}{}_{\hat{\mu}\hat{\nu}} \ . \tag{18}$$

Combining these last two equations yields

$$\Gamma_{\hat{\alpha}\hat{\mu}\hat{\nu}} = \frac{1}{2}\left(\omega_{\hat{\mu}\hat{\alpha}\hat{\nu}} + \omega_{\hat{\nu}\hat{\alpha}\hat{\mu}} - \omega_{\hat{\alpha}\hat{\mu}\hat{\nu}}\right) \quad \text{in an orthonormal basis.} \tag{19}$$

The connection coefficients in an orthonormal basis are also called Ricci rotation coefficients (Wald) or the spin connection (Carroll).

It is straightforward to generalize the results of this section to general bases that are neither orthonormal nor coordinate. The commutator basis coefficients are defined as in equation (12). Dropping the carets on the indices, the general connection is (MTW eq. 8.24b)

$$\Gamma_{\alpha\mu\nu} \equiv g_{\alpha\beta}\Gamma^{\beta}{}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu} + \omega_{\mu\alpha\nu} + \omega_{\nu\alpha\mu} - \omega_{\alpha\mu\nu}\right) \quad \text{in any basis.} \tag{20}$$

The results for coordinate bases (where $\omega_{\alpha\mu\nu} = 0$) and for orthonormal bases (where $\partial_\alpha g_{\mu\nu} = 0$) follow as special cases.

3 Parallel transport and geodesics

3.1 Differentiation along a curve

As a prelude to parallel transport we consider another form of differentiation: differentiation along a curve. A curve is a parametrized path through spacetime: $\mathbf{x}(\lambda)$, where λ is a parameter that varies smoothly and monotonically along the path. The curve has a tangent vector $\vec{V} \equiv d\vec{x}/d\lambda = (dx^\mu/d\lambda)\,\vec{e}_\mu$. Here one must be careful about the interpretation: x^μ are not the components of a vector; they are simply 4 scalar fields. However, $\vec{V} = d\vec{x}/d\lambda$ is a vector (i.e. a tangent vector in the manifold).

If we wish, we could make $\vec{V}$ a unit vector (provided $\vec{V}$ is non-null) by setting $d\lambda = |d\vec{x}\cdot d\vec{x}\,|^{1/2}$ to measure path length along the curve. However, we will impose no such restriction in general.

Now, suppose that we have a scalar field $f_{\mathbf{x}}$ defined along the curve. We define the derivative along the curve by a simple extension of equations (36) and (38) of the first set of lecture notes:

$$\frac{df}{d\lambda} \equiv \nabla_V f \equiv \langle \tilde{\nabla} f, \vec{V} \rangle = V^\mu \partial_\mu f \ , \quad \vec{V} = \frac{d\vec{x}}{d\lambda} . \tag{21}$$

We have introduced the symbol ∇_V for the **directional derivative**, i.e. the covariant derivative along $\vec{V}$, the tangent vector to the curve $\mathbf{x}(\lambda)$. This is a natural generalization of ∇_μ, the covariant derivative along the basis vector $\vec{e}_\mu$.

For the derivative of a scalar field, ∇_V involves just the partial derivatives ∂_μ. Suppose, however, that we differentiate a vector field $\vec{A}_{\mathbf{x}}$ along the curve. Now the components of the gradient $\nabla_\mu A^\nu$ are not simply the partial derivatives but also involve the connection. The same is true when we project the gradient onto the tangent vector $\vec{V}$ along a curve:

$$\frac{d\vec{A}}{d\lambda} \equiv \frac{DA^\mu}{d\lambda}\,\vec{e}_\mu \equiv \nabla_V \vec{A} \equiv \langle \tilde{\nabla}\vec{A}, \vec{V} \rangle = V^\nu (\nabla_\nu A^\mu)\,\vec{e}_\mu = \left(\frac{dA^\mu}{d\lambda} + \Gamma^\mu{}_{\kappa\nu} A^\kappa V^\nu \right) \vec{e}_\mu \ . \tag{22}$$

We retain the symbol ∇_V to indicate the covariant derivative along $\vec{V}$ but we have introduced the new notation $D/d\lambda = V^\mu \nabla_\mu \neq d/d\lambda = V^\mu \partial_\mu$.

3.2 Parallel transport

The derivative of a vector along a curve leads us to an important concept called parallel transport. Suppose that we have a curve $\mathbf{x}(\lambda)$ with tangent $\vec{V}$ and a vector $\vec{A}(0)$ defined at one point on the curve (call it $\lambda = 0$). We define a procedure called parallel transport by defining a vector $\vec{A}(\lambda)$ along each point of the curve in such a way that $DA^\mu/d\lambda = 0$:

$$\nabla_V \vec{A} = 0 \quad \Leftrightarrow \quad \text{parallel transport of } \vec{A} \text{ along } \vec{V}. \tag{23}$$

Over a small distance interval this procedure is equivalent to transporting the vector $\vec{A}$ along the curve in such a way that the vector remains parallel to itself with constant length: $\vec{A}(\lambda + \Delta\lambda) = \vec{A}(\lambda) + O(\Delta\lambda)^2$. In a locally flat coordinate system, with the connection vanishing at $\mathbf{x}(\lambda)$, the components of the vector do not change as the vector is transported along the curve. If the space were globally flat and we used rectilinear coordinates (with vanishing connection everywhere), the components would not change at all no matter how the vector is transported. This is not the case in a curved space or in a flat space with curvilinear coordinates because in these cases the connection does not vanish everywhere.

3.3 Geodesics

Parallel transport can be used to define a special class of curves called *geodesics*. A geodesic curve is one that parallel-transports its own tangent vector $\vec{V} = d\vec{x}/d\lambda$, i.e., a curve that satisfies $\nabla_V \vec{V} = 0$. In other words, not only is $\vec{V}$ kept parallel to itself (with constant magnitude) along the curve, but locally the curve continues to point in the same direction all along the path. A geodesic is the natural extension of the definition of a "straight line" to a curved manifold. Using equations (22) and (23), we get a second-order differential equation for the coordinates of a geodesic curve:

$$\frac{DV^\mu}{d\lambda} = \frac{dV^\mu}{d\lambda} + \Gamma^\mu{}_{\alpha\beta} V^\alpha V^\beta = 0 \quad \text{for a geodesic} \ , \quad V^\mu \equiv \frac{dx^\mu}{d\lambda} \ . \tag{24}$$

Indeed, in locally flat coordinates (such that the connection vanishes at a point), this is the equation of a straight line. However, in a curved space the connection cannot be made to vanish everywhere. A well-known example of a geodesic in a curved space is a great circle on a sphere.

There are several technical points worth noting about geodesic curves. The first is that $\vec{V} \cdot \vec{V} = \mathsf{g}(\vec{V}, \vec{V})$ is constant along a geodesic because $d\vec{V}/d\lambda = 0$ (eq. 24) and $\nabla_V \mathsf{g} = 0$ (metric compatibility with gradient). Therefore, a geodesic may be classified by its tangent vector as being either timelike ($\vec{V} \cdot \vec{V} < 0$), spacelike ($\vec{V} \cdot \vec{V} > 0$) or null ($\vec{V} \cdot \vec{V} = 0$). The second point is that a nonlinear transformation of the parameter λ will invalidate equation (24). In other words, if $x^\mu(\lambda)$ solves equation (24), $y^\mu(\lambda) \equiv x^\mu(\xi(\lambda))$ will not solve it unless $\xi = a\lambda + b$ for some constants a and b. Only a special class of parameters, called *affine parameters*, can parametrize geodesic curves.

The affine parameter has a special interpretation for a non-null geodesic. We deduce this relation from the constancy along the geodesic of $\vec{V} \cdot \vec{V} = (d\vec{x} \cdot d\vec{x})/(d\lambda^2) \equiv a$, implying $ds = a d\lambda$ and therefore $s = a\lambda + b$ where s is the path length ($ds^2 = g_{\mu\nu} dx^\mu dx^\nu$). For a non-null geodesic ($\vec{V} \cdot \vec{V} \neq 0$), all affine parameters are linear functions of path length (or proper time, if the geodesic is timelike). The linear scaling of path length amounts simply to the freedom to change units of length and to choose any point as $\lambda = 0$. Note that originally we imposed no constraints on the parameterization. However, the solutions of the geodesic equation automatically have λ being an affine parameter. There is no fundamental reason to use an affine parameter; one could always take a solution of the geodesic equation and reparameterize it or eliminate the parameter altogether by replacing it with one of the coordinates along the geodesic. For example, for a timelike trajectory, $x^i(t)$ is a perfectly valid description and is equivalent to $x^\mu(\lambda)$. But the spatial components as functions of $t = x^0$ clearly do not satisfy the geodesic equation for $x^\mu(\lambda)$.

Another interesting point is that the total path length is stationary for a geodesic:

$$\delta \int_A^B ds = \delta \int_A^B \left| g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \right|^{1/2} d\lambda = 0 \tag{25}$$

if λ is an affine parameter. The δ refers to a variation of the integral arising from a variation of the curve, $x^\mu(\lambda) \to x^\mu(\lambda) + \delta x^\mu(\lambda)$, with fixed endpoints. The metric components are considered here to be functions of the coordinates. The variational principle is discussed in section 2 of the 8.962 notes "Hamiltonian Dynamics of Particle Motion," where it is shown that stationary path length implies the geodesic equation (24) if the parameterization is affine. Equation (25) is invariant under reparameterization, so its stationary solutions are a broader class of functions than the solutions of equation (24). In general, the tangent vector of the stationary solutions are not normalized: $|\vec{V} \cdot \vec{V}|^{1/2} = Q(\lambda) \neq$ constant, implying that λ is not affine. It is easy to show that any stationary solution may be reparameterized, $\lambda \to \tau$ through $d\tau/d\lambda = Q(\lambda)$, and that the resulting curve $x^\mu(\lambda(\tau))$ obeys the geodesic equation with affine parameter τ. This transformation replaces the unnormalized tangent vector $\vec{V}$ by $\vec{V}/Q(\lambda)$. For an affine parameterization, the tangent vector must always have constant length.

Equation (25) is a curved space generalization of the statement that a straight line is the shortest path between two points in flat space.

3.4 Integrals of motion and Killing vectors

Equation (24) is a set of four second-order nonlinear ordinary differential equations for the coordinates of a geodesic curve. One may ask whether the order of this system can be reduced by finding integrals of the motion. An integral, also called a conserved quantity, is a function of x^μ and $V^\mu = dx^\mu/d\lambda$ that is constant along any geodesic. At least one integral always exists: $\vec{V} \cdot \vec{V} = g_{\mu\nu} V^\mu V^\nu$. (For an affine parameterization, $\vec{V} \cdot \vec{V}$ is constant along the curve.) Are there others? Sometimes. One may show that equation (24) may be rewritten as an equation of motion for $V_\mu \equiv g_{\mu\nu} V^\nu$, yielding

$$\frac{dV_\mu}{d\lambda} = \frac{1}{2} (\partial_\mu g_{\alpha\beta}) V^\alpha V^\beta \ . \tag{26}$$

Consequently, if all of the metric components are independent of some particular coordinate x^μ, the corresponding component of the tangent one-form is constant along the geodesic. This result is very useful in reducing the amount of integration needed to construct geodesics for metrics with high symmetry. However, the condition $\partial_\mu g_{\alpha\beta} = 0$ is coordinate-dependent. There is an equivalent coordinate-free test for integrals, based on the existence of special vector fields $\vec{K}$ call *Killing vectors*. Killing vectors are, by definition, solutions of the differential equation

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \ . \tag{27}$$

(The Killing *vector* components are, of course, $K^\mu = g^{\mu\nu} K_\nu$.) The Killing equation (27) usually has no solutions, but for highly symmetric spacetime manifolds there may be one or more solutions. It is a nice exercise to show that each Killing vector leads to the integral of motion

$$\langle \tilde{V}, \vec{K} \rangle = K^\mu V_\mu = \text{constant along a geodesic} . \tag{28}$$

Note that if one of the basis vectors (for some basis) satisfies the Killing equation, then the corresponding component of the tangent one-form is an integral of motion. The test for integrals implied by equation (26) is a special case of the Killing vector test when the Killing vector is simply a coordinate basis vector.

The discussion here has focused on geodesics as curves. The notes "Hamiltonian Dynamics of Particle Motion" interprets them as worldlines for particles because, as we will see, a fundamental postulate of general relativity is that, in the absence of non-gravitational forces, particles move along geodesics. Given this fact, we are free to choose units of the affine parameter λ so that $dx^\mu/d\lambda$ is the 4-momentum P^μ, normalized by $\vec{P} \cdot \vec{P} = -m^2$ for a particle of mass m (instead of $dx^\mu/d\lambda = V^\mu$, $\vec{V} \cdot \vec{V} = -1$). Thus, the tangent vector, denoted $\vec{V}$ above, is equivalent to the particle 4-momentum vector. The affine parameter λ then measures proper time divided by particle mass. Although one might fear this makes no sense for a massless particle, in fact it is the only way to affinely parameterize null geodesics because the proper time change $d\tau$ vanishes along a null geodesic so $dx^\mu/d\tau$ is undefined. For a massless particle, one takes the limit $m \to 0$ starting from the solution for a massive particle, with the result that $d\lambda = d\tau/m$ is finite as $m \to 0$.

4 Curvature

We introduce curvature by considering parallel transport around a general (non-geodesic) closed curve. In flat space, in a globally flat coordinate system (for which the connection vanishes everywhere), parallel transport leaves the components of a vector unchanged. Thus, in flat space, transporting a vector around a closed curve returns the vector to its starting point unchanged. Not so in a nonflat space. This change under a closed cycle is called an "anholonomy."

Consider, for example, a sphere. Suppose that we have a vector pointing east on the equator at longitude 0°. We parallel transport the vector eastward on the equator by 180°. At each point on the equator the vector points east. Now the vector is parallel transported along a line of constant longitude over the pole and back to the starting point on the equator. At each point on this second part of the curve, the vector points at right angles to the curve, and its direction never changes. Yet, at the end of the curve, at the same point where the curve started, the vector points west!

The reader may imagine that the example of the sphere is special because of the sharp changes in direction made in the path. However, parallel transport around any

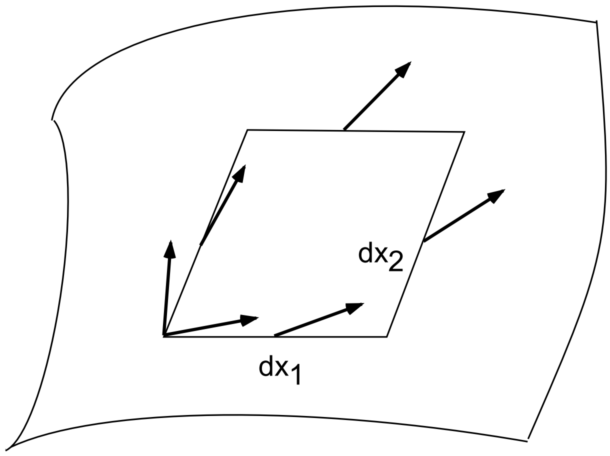

Figure 1: Parallel transport around a closed curve. The vector in the lower-left corner is parallel transported in a counter-clockwise direction along around 4 segments $d\vec{x}_1$, $d\vec{x}_2$, $-d\vec{x}_1$, and $-d\vec{x}_2$. At the end of the journey, the vector has been rotated. This mismatch ("anholonomy") does not occur for parallel transport in a flat space; its existence is the defining property of curvature.

smooth closed curve results in an anholonomy on a sphere. For example, consider a latitude circle away from the equator. Imagine you are an airline pilot flying East from Boston. If you were flying on a great circle route, you would soon be flying in a south-east direction. If you parallel transport a vector along a geodesic, its direction relative to the tangent vector (direction of motion) does not change, i.e. $\nabla_V(\vec{A}\cdot\vec{V}\,) = 0$ for parallel transport of $\vec{A}$ along tangent $\vec{V}$. Parallel transport implies $\nabla_V\vec{A} = 0$; moreover, $\nabla_V\vec{V} = 0$ for a geodesic. However, a constant-latitude circle is not a geodesic, hence $\nabla_V\vec{V} \neq 0$. In order to maintain a constant latitude, you will have to constantly steer the airplane north compared with a great circle route. Consequently, the angle between $\vec{A}$ (which is parallel-transported) and the tangent changes: $\nabla_V(\vec{A}\cdot\vec{V}\,) = A\cdot(\nabla_V\vec{V}\,)$. A nonzero rotation accumulates during the trip, leading to a net rotation of $\vec{A}$ around a closed curve.

We can refine this into a definition of curvature as follows. Suppose that our closed curve consists of four infinitesimal segments: $d\vec{x}_1$, $d\vec{x}_2$, $-d\vec{x}_1$ and $-d\vec{x}_2$. In a flat space this would be called a parallelogram and the difference $d\vec{A}$ between the final and initial vectors would vanish. In a curved space we can create a parallelogram by taking two pairs of coordinate lines and choose $d\vec{x}_1$ and $d\vec{x}_2$ to point along the coordinate lines (e.g. in directions $\vec{e}_1$ and $\vec{e}_2$). Parallel transport around a closed curve gives a change in the vector $d\vec{A}$ that must be proportional to $\vec{A}$, to $d\vec{x}_1$, and to $d\vec{x}_2$. Remarkably, it is proportional to nothing else. Therefore, $d\vec{A}$ is given by a rank $(1,3)$ tensor called the Riemann curvature tensor:

$$d\vec{A}(\,\cdot\,) \equiv -\mathsf{R}(\,\cdot\,,\vec{A},d\vec{x}_1,d\vec{x}_2) = -\vec{e}_\mu R^\mu{}_{\nu\alpha\beta}\,A^\nu\,dx_1^\alpha\,dx_2^\beta\ . \tag{29}$$

The dots indicate that a one-form is to be inserted; recall that a vector is a function of a one-form. The minus sign is purely conventional and is chosen for agreement with MTW. Note that the Riemann tensor must be antisymmetric on the last two slots because reversing them amounts to changing the direction around the parallelogram, i.e. swapping the final and initial vectors $\vec{A}$, hence changing the sign of $d\vec{A}$.

All standard GR textbooks show that equation (29) is equivalent to the following important result known as the **Ricci identity**

$$(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)A^\mu = R^\mu{}_{\nu\alpha\beta}A^\nu \quad \text{in a coordinate basis}\ . \tag{30}$$

In a non-coordinate basis, there is an additional term on the left-hand side, $-\nabla_C A^\mu$ where $\vec{C} \equiv [\vec{e}_\alpha,\vec{e}_\beta\,]$. This commutator vanishes for a coordinate basis (eq. 12).

Equation (30) is a remarkable result. In general, there is no reason whatsoever that the derivatives of a vector field should be related to the vector field itself. Yet the difference of second derivatives is not only related to, but is linearly proportional to the vector field! This remarkable result is a mathematical property of metric spaces with connections. It is equivalent to the statement that parallel transport around a small closed parallelogram is proportional to the vector and the oriented area element (eq. 29).

Equation (30) is similar to equation (11). The torsion tensor and Riemann tensor are geometric objects from which one may build a theory of gravity in curved spacetime. In general relativity, the torsion is zero and the Riemann tensor holds all of the local information about gravity.

It is straightforward to determine the components of the Riemann tensor using equation (30) with $\vec{A} = \vec{e}_\nu$. The result is

$$R^\mu{}_{\nu\alpha\beta} = \partial_\alpha\Gamma^\mu{}_{\nu\beta} - \partial_\beta\Gamma^\mu{}_{\nu\alpha} + \Gamma^\mu{}_{\kappa\alpha}\Gamma^\kappa{}_{\nu\beta} - \Gamma^\mu{}_{\kappa\beta}\Gamma^\kappa{}_{\nu\alpha} \quad \text{in a coordinate basis}\ . \tag{31}$$

Note that some authors (e.g., Weinberg 1972) define the components of Riemann with opposite sign. Our sign convention follows Misner et al (1973), Wald (1984) and Schutz (1985).

Note that the Riemann tensor involves the first and second partial derivatives of the metric (through the Christoffel connection in a coordinate basis). Weinberg (1972) shows that the Riemann tensor is the only tensor that can be constructed from the metric tensor and its first and second partial derivatives and is linear in the second derivatives. Recall that one can always define locally flat coordinates such that $\Gamma^{\mu}{}_{\nu\lambda} = 0$ at a point. However, one cannot choose coordinates such that $\Gamma^{\mu}{}_{\nu\lambda} = 0$ everywhere unless the space is globally flat. The Riemann tensor vanishes everywhere if and only if the manifold is globally flat. This is a very important result.

If we lower an index on the Riemann tensor components we get the components of a $(0,4)$ tensor:

$$R_{\mu\nu\kappa\lambda} = g_{\mu\alpha}R^{\alpha}{}_{\nu\kappa\lambda} = \frac{1}{2}\left(g_{\mu\lambda,\nu\kappa} - g_{\mu\kappa,\nu\lambda} + g_{\nu\kappa,\mu\lambda} - g_{\nu\lambda,\mu\kappa}\right) + g_{\alpha\beta}\left(\Gamma^{\alpha}{}_{\mu\lambda}\Gamma^{\beta}{}_{\nu\kappa} - \Gamma^{\alpha}{}_{\mu\kappa}\Gamma^{\beta}{}_{\nu\lambda}\right) \ , \tag{32}$$

where we have used commas to denote partial derivatives for brevity of notation: $g_{\mu\lambda,\nu\kappa} \equiv \partial_\kappa\partial_\nu g_{\mu\lambda}$. In this form it is easy to determine the following symmetry properties of the Riemann tensor:

$$R_{\mu\nu\kappa\lambda} = R_{\kappa\lambda\mu\nu} = -R_{\nu\mu\kappa\lambda} = -R_{\mu\nu\lambda\kappa} \ , \quad R_{\mu\nu\kappa\lambda} + R_{\mu\kappa\lambda\nu} + R_{\mu\lambda\nu\kappa} = 0 \ . \tag{33}$$

It can be shown that these symmetries reduce the number of independent components of the Riemann tensor in four dimensions from 4^4 to 20.

4.1 Bianchi identities, Ricci tensor and Einstein tensor

We note here several more mathematical properties of the Riemann tensor that are needed in general relativity. First, by differentiating the components of the Riemann tensor one can prove the *Bianchi identities*:

$$\nabla_\sigma R^{\mu}{}_{\nu\kappa\lambda} + \nabla_\kappa R^{\mu}{}_{\nu\lambda\sigma} + \nabla_\lambda R^{\mu}{}_{\nu\sigma\kappa} = 0 \ . \tag{34}$$

Note that the gradient symbols denote the covariant derivatives and not the partial derivatives (otherwise we would not have a tensor equation). The Bianchi identities imply the vanishing of the divergence of a certain $(2,0)$ tensor called the Einstein tensor. To derive it, we first define a symmetric contraction of the Riemann tensor, known as the Ricci tensor:

$$R_{\mu\nu} \equiv R^{\alpha}{}_{\mu\alpha\nu} = R_{\nu\mu} = \partial_\kappa\Gamma^{\kappa}{}_{\mu\nu} - \partial_\mu\Gamma^{\kappa}{}_{\kappa\nu} + \Gamma^{\kappa}{}_{\kappa\lambda}\Gamma^{\lambda}{}_{\mu\nu} - \Gamma^{\kappa}{}_{\mu\lambda}\Gamma^{\lambda}{}_{\kappa\nu} \ . \tag{35}$$

One can show from equations (33) that any other contraction of the Riemann tensor either vanishes or is proportional to the Ricci tensor. The contraction of the Ricci tensor is called the Ricci scalar:

$$R \equiv g^{\mu\nu}R_{\mu\nu} \ . \tag{36}$$

Contracting the Bianchi identities twice and using the antisymmetry of the Riemann tensor one obtains the following relation:

$$\nabla_\nu G^{\mu\nu} = 0 \ , \quad G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R = G^{\nu\mu} \ . \tag{37}$$

The symmetric tensor $G^{\mu\nu}$ that we have introduced is called the *Einstein tensor*. Equation (37) is a mathematical identity, not a law of physics. Through the Einstein equations it provides a deep illustration of the connection between mathematical symmetries and physical conservation laws.

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