

8.08 Statistical Physics II

Bose - Einstein condensation
(Dated: April 4, 2011)

1. Examine a gas of non-interacting spin 1 particles with energy $H = \frac{p^2}{2m} - \mu_0 s_z B$

Solution: The occupation number of each state is

$$n_s = \frac{1}{e^{\beta p^2/2m - \beta \mu_0 s B - \beta \mu} - 1}$$

The total number of particles with each spin is

$$N_s = \frac{V}{(2\pi)^3} \int d^3k \frac{1}{e^{\beta p^2/2m - \beta \mu_0 s B - \beta \mu} - 1} = \frac{V}{\lambda^3} f_{3/2}(ze^{\beta \mu_0 s B})$$

where $f_m(z) = \frac{1}{\Gamma(m)} \int_0^\infty \frac{dx x^{m-1}}{z^{-1}e^x - 1}$, $\lambda = h/\sqrt{2\pi m k_B T}$ and $z = e^{\beta \mu}$

The magnetization is

$$M = \mu_0(N_+ - N_-) = \mu_0 \frac{V}{\lambda^3} [f_{3/2}(ze^{\beta \mu_0 B}) - f_{3/2}(ze^{-\beta \mu_0 B})]$$

For small magnetic field we have the expansion

$$f_{3/2}(ze^{\beta \mu_0 s B}) \approx f_{3/2}(1 + s\beta \mu_0 B) + sz\beta \mu_0 B \frac{\partial}{\partial z} f_{3/2}(z)$$

So

$$M = \frac{2\mu_0^2 V}{k_B T \lambda^3} B f_{1/2}(z)$$

and

$$\chi = \left(\frac{\partial M}{\partial B} \right)_{B=0} = \frac{2\mu_0^2 V}{k_B T \lambda^3} B f_{1/2}(z)$$

Bose-Einstein condensation happens when

$$n = \frac{3}{\lambda^3} f_{3/2}(1)$$

which leads at a temperature

$$T_c(n) = \frac{h^2}{2\pi m k_B} \left(\frac{n}{3\zeta_{3/2}} \right)^{2/3}$$

where $\zeta_{3/2} = f_{3/2}(1) \approx 2.61$ and the susceptibility diverges.

2. Calculate the critical temperature of a non-interacting non-relativistic boson gas in a cubic box of length L

Solution:

$$N - N_0 \approx \int_0^\infty ndN$$

The non relativistic energy for a particle in a box is

$$E_n = \frac{\hbar^2 \pi^2}{2mL^2} (j^2 + k^2 + l^2) = E(j^2 + k^2 + l^2)$$

The differential number of states is

$$dN = \frac{L^3}{(2\pi\hbar)^3} |p|^2 dp d\Omega = \frac{L^3}{(2\pi\hbar)^3} \sqrt{2m^3 E} dE d\Omega = \frac{\sqrt{E}}{16 \left(\frac{L^3}{(2\pi\hbar)^3}\right)^{3/2}} dE d\Omega$$

The number of excited states is

$$N - N_0 = \int ndN = \frac{4\pi}{16} \int_0^\infty dE \frac{\sqrt{E}}{\left(\frac{\hbar^2 \pi^2}{2mL^2}\right)^{3/2}} \frac{1}{e^{\beta E - \beta \mu} - 1}$$

Close to the transition

$$N - N_0 = \left(\frac{\pi k T}{4E}\right)^{3/2} \zeta(3/2)$$

Finally the number of particles in the ground state are

$$N_0 = N \left[1 - T^{3/2} \left(\frac{4E}{\pi k}\right)^{-3/2} \left(\frac{\zeta^{3/2}(3/2)}{N^{2/3}}\right)^{3/2}\right] = N \left[1 - (T/T_c)^{3/2}\right]$$

3. Examine a non interacting boson gas in a three dimensional harmonic potential

Solution:

The energy of each state is

$$E_n = \hbar\omega(n_x + n_y + n_z) = \hbar\omega q$$

where we have neglected the zero point energy.

$$N - N_0 = \sum \frac{0.5(q+1)(q+2)}{e^{\beta(q\hbar\omega - \mu)} - 1} \approx \int_0^\infty \frac{0.5(q^2 + 3q) dq}{e^{\beta(q\hbar\omega - \mu)} - 1}$$

The occupation of the ground state is to lowest order

$$N_0 = N \left[1 - (T/T_c)^3\right]$$

where

$$T_c = \frac{\hbar\omega}{k} \left(\frac{N}{\zeta(3)}\right)^{1/3}$$