

8.08 Statistical Physics II

Fermions

(Dated: April 21, 2011)

1. Assume that the inside of a metal represents a square potential well for electrons. Starting from Pauli's principle, derive the velocity distribution of the electrons as a function of temperature. Derive the formula for the current in a vacuum diode with planar electrodes for negative plate voltage.

Solution:

The density of particles in velocity space is

$$n(v)d^3v = \frac{2m^3}{h^3} d^3v \frac{1}{e^{(mv^2/2 - E_F)/kT} + 1}$$

The current density is

$$J = e \langle nv_x \rangle = 2e \left(\frac{m}{h}\right)^3 \int_{-\infty}^{\infty} dv_y dv_z \int_u^{\infty} \frac{v_x dv_x}{e^{(mv^2/2 - E_F)/kT} + 1}$$

where $mu^2/2 = E_F + \Phi + eV$

Assuming $e(\Phi + V) \gg kT$ we have

$$J = 2e \left(\frac{m}{h}\right)^3 e^{E_F/kT} \int_u^{\infty} v_x e^{-mv_x^2/2kT} \int_{-\infty}^{\infty} dv_y dv_z e^{-m(v_y^2 + v_z^2)/2kT} = 4\pi em \frac{k^2 T^2}{h^3} e^{-(\Phi + eV)/kT}$$

2. Compute the ratio of thermal conductivity of helium gas at $P = 0.1atm$ and $300K$ to that at $P = 0.5atm$ and $300K$. Compute also the ratio of viscosities at the two pressures

Solution: The rate of energy transfer through a unit area in the plane $Z = const$ is proportional to

$$(nv)(-\lambda dE/dT) = (nv\lambda dE/dT)(-dT/dZ)$$

The first factor is the flux of molecules in either direction across the plane. The second term is the difference in average molecular energy in a distance of one mean free path (λ). The coefficient of $(-dT/dZ)$ is the heat conductivity, $K \propto nv\lambda C$, where C is the molecular specific heat.

The mean free path is $\lambda = 1/n\sigma$, where σ is the collision cross section. In addition, $C = 3k/2$ for a monoatomic gas and $v = \sqrt{3kT/M}$. Finally

$$K \propto \sqrt{T}$$

independent of pressure, so the desired ratio is 1.

The viscosity may be calculated by considering the net transverse momentum transfer across a unit area of a plane $Z = const$, in unit time. In the same notation, this is proportional to $-(nv)(m\lambda dU/dZ)$, where m is the molecular mass and u is the transverse velocity. By definition, the coefficient of du/dZ is the viscosity η . Thus

$$\eta \propto \sqrt{T}$$

independent of pressure, so the ratio is 1. Furthermore

$$\eta/K = const$$

3. A small circular opening of radius a which is small compared to the mean free path of mercury, is cut in the wall of a very thin walled rectangular tank containing mercury vapor at temperature T and very low pressure P . At a distance h above the hole and parallel to the wall of the tank, a collecting sheet of metal is placed and cooled so that when any of the mercury atoms strike it, the vapor condenses at once. Derive an expression for the distribution of mercury in gm/cm^3 on the collecting sheet at time t in terms of polar angle θ between the normal to the hole and the point of collection.

Solution:

$dN(v) = N_0 \rho(v) d^3v$ where $\rho(v) = (m/2\pi kT)^{3/2} \exp(-mv^2/2kT)$ and N_0 is the density of mercury atoms, $N_0 = P/kT$.

At time t , the number of atoms with speed v which have escaped into the solid angle $d\Omega$ at an angle θ is

$$dn = (\pi a^2 \cos(\theta))(vt) N_0 \rho(v) v^2 dv d\Omega$$

Of these the number $dn' = (\pi a^2 \cos(\theta))(vt)r N_0 \rho(v) v^2 dv d\Omega$ are still in flight and have not struck the collector. In addition the speed must satisfy $vt > r$ or none of these will have struck the collector ($r = h/\cos\theta$). Hence, in time t , the total number striking the collector in the solid angle $d\Omega$ is

$$dn(t) = (\pi a^2 \cos(\theta)) N_0 d\Omega \int_{r/t}^{\infty} dv v^2 \rho(v) (vt - r)$$

After a little manipulation, the integral is shown to be

$$f(r, t) \equiv \int_{r/t}^{\infty} dv v^2 \rho(v) (vt - r) = (\bar{v}t/4\pi) \exp(-4r^2/\pi v^2 t^2) - (r/4\pi) E(2r/\bar{v}t\pi^{1/2})$$

where $\bar{v} = 4\pi \int_0^{\infty} v^3 \rho(v) dv = \sqrt{8kT/\pi m}$ is the mean speed, and $E(x)$ is the error function defined as

$$E(x) = (2/\sqrt{\pi}) \int_x^{\infty} dy e^{-y^2} \text{ with } E(0) = 1.$$

For $\bar{v}t \gg r$ $f(r, t)$ reduces to $(\bar{v}t - r)/4\pi$. Finally $d\Omega = d\sigma \cos\theta / r^2$.

The mass collected per unit area in time t is given by

$$\frac{dM}{d\sigma} = \frac{\pi m P a^2 \cos^4 \theta}{h^2 k T} f(h/\cos\theta, t)$$