8.08 Statistical Physics II

Fluctuations
(Dated: March 8, 2011)

1. The Helmholtz free energy A of a ferromagnet can be assumed to depend on the magnetization for no external field as follows

$$A = A_0 + \alpha(T - T_c)M^2 + \beta M^4$$

for temperature T near the Curie temperature T_c; α and β are positive and approximately temperature-independent. A_0 is the free energy of the unmagnetized state, and may be regarded as approximately temperature independent. Derive the temperature dependence of the average magnetization $M(T)$ which follows from the above equation if fluctuations are neglected. What is the influence of fluctuations $M(T)$? Obtain also the magnetic susceptibility above the Curie temperature

Solution:

The average magnetization is

$$\bar{M} = \frac{\int_0^\infty dM \; M e^{-A/kT}}{\int_0^\infty dM \; e^{-A/kT}}$$

When one neglects fluctuations, this is equal to the result we get by mimimizing the free energy A, so $M_0 = \left[\frac{\alpha}{2\beta}(T_c - T)\right]^{1/2}$ for $T < T_c$ and zero for $T > T_c$

To see the effect of fluctuations we consider the cases (a) $T < T_c$ and (b) $T > T_c$.

(a) Let's define a new variable x that determines the deviation from M_0.

$$M = M_0 + x\left[\frac{kT}{2\alpha(T_c - T)}\right]^{1/2}$$

where $M_0 = \left[\frac{\alpha}{2\beta}(T_c - T)\right]^{1/2}$

Expanding the free energy around M_0 we get

$$A = A_0 - \frac{\alpha^2}{4\beta}(T_c - T)^2 + kT(x^2 + x^3/B + x^4/4B^2)$$

where $B = [\alpha^2(T_c - T)^2/\beta kT]^{1/2}$.

We now get

$$\frac{\bar{M} - M_0}{M_0} = \frac{\int_{-B}^\infty dx \; x e^{-(x^2+x^3/B+x^4/4B^2)}}{\int_{-B}^\infty dx \; e^{-(x^2+x^3/B+x^4/4B^2)}}$$

In the $B \gg 1$ limit, we can approximate the above integrals by

$$\frac{\bar{M} - M_0}{M_0} = \frac{-\int_{-\infty}^\infty dx \; x^4 e^{-x^2}}{\int_{-\infty}^\infty dx \; e^{-x^2}} = -\frac{3}{4B^2} = -\frac{3\beta kT}{4\alpha^2(T_c - T)^2}$$

(b) We define

$$M = y\left[\frac{kT}{\alpha(T-T_c)}\right]^{1/2}$$

By plugging this into the free energy we get

$$A = A_0 + kT(y^2 + y^4/B^2)$$

where B is the same as before. The average magnetization is

$$\bar{M} = \frac{\int_0^\infty dy\ ye^{-(y^2+y^4/B^2)}}{\int_0^\infty dy\ e^{-(y^2+y^4/B^2)}} \approx \left[\frac{kT}{\pi\alpha(T-T_c)}\right]$$

in the $B \gg 1$ limit.
In this regime the response on a magnetic field is important. So we add the term $-M \cdot H$ to the free energy. This leads to

$$\bar{M}(H) = \left[\frac{kT}{\alpha(T-T_c)}\right] \frac{\int_0^\infty dy\ ye^{-(y^2+yH/[kT\alpha(T-T_c)]^{1/2})}}{\int_0^\infty dy\ e^{-(y^2)}} \approx \frac{H}{2\alpha(T-T_c)}$$

where terms of order $1/B \ll 1$ have been neglected.
The above result is known as the Curie-Weiss law. The susceptibility is

$$\chi = \left(\frac{\partial \bar{M}}{\partial H}\right)_{H=0} = \frac{1}{2\alpha(T-T_c)}$$