8.08 Statistical Physics II

Fluctuations and BE statistics
(Dated: March 14, 2011)

1. A wire of length l and mass per unit length μ is fixed at both ends and tightened to a tension τ. What is the rms fluctuation, in classical statistics, of the midpoint of the wire when it is in equilibrium with a heat bath at temperature T? A useful series is $\sum_0^\infty (2m+1)^{-2} = \frac{\pi^2}{8}$

   Solution:

   The equation of motion of the string is

$$\frac{d^2y}{dt^2} = \frac{\tau}{\mu}\frac{d^2y}{dx^2}$$

   The solutions are the linear combinations of

$$y(x,t) = A_n sin(\omega_n t + \alpha) sin(k_n x)$$

   where$\frac{\omega_n^2}{k_n^2} = \frac{\tau}{\mu}$ and $k_n = \frac{n\pi}{l}$

   Only the odd modes contribute to the fluctuation of the midpoint of the midpoint. The kinetic energy of each mode is $\epsilon_n = \int_0^l \frac{1}{2}\left(\frac{dy}{dt}\right)^2 dx$ and the total energy is the maximum over time of the kinetic energy of the mode. So

$$E_n = \frac{\tau n^2 \pi^2}{4l} A_n^2$$

   In thermal equilibrium $E_n = k_B T$. Solving for the amplitude of each mode, we get

$$A_n = \frac{4lk_B T}{\tau \pi^2 n^2}$$

   Finally

$$\langle y^2 \rangle = \langle (\sum A_n sin(\omega_n t + \alpha) sin(n\pi/2))^2 \rangle = \frac{1}{2}\sum A_n^2 = \frac{4lk_B T}{2\tau\pi^2}\sum \frac{1}{n^2} \Rightarrow y_{rms} = \frac{1}{2}\sqrt{\frac{lk_B T}{\tau}}$$

2. The lattice specific heat of a certain form of carbon has a temperature dependence t^2, instead of the more common T^3 dependence for solids. What can you infer about the structure of this particular phase of carbon?

   Solution:

   The energy due to lattice vibrations is

$$U = \int \frac{\hbar\omega g(\omega) d\omega}{e^{\hbar\omega/kT} - 1}$$

   where $g(\omega)$ is the density of states of phonons. At low temperatures, for a three dimensional lattice is

$$g(\omega)d\omega = \frac{d^3k}{(2\pi)^3} = \frac{4\pi k^2 dk}{(2\pi)^2} = \frac{4\pi\omega^2 d\omega}{(2\pi)^3 c^3}$$

   . So

$$g(\omega) \sim \omega^2 \Rightarrow E \sim T^4 \Rightarrow C \sim T^3$$

   If the solid is composed of two-dimensional crystals (like graphite)

$$g(\omega) \sim \omega \Rightarrow C \sim T^2$$

3. The surface temperature of the sun is $T_0 = 5500K$, its radius $R = 7 \cdot 10^10cm$ while earth's radius is $r = 6.37 \cdot 10^8 cm$. The mean distance is $L = 1.5 \cdot 10^13cm$. Find an approximate expression for earth's temperature, assuming both are black-bodies in equilibrium.

   Solution: The power given off by the sun is

$$P = \sigma T_0^4 (4\pi R^2)$$

   The power that the earth absorbs is

$$P_i = \sigma T_0^4 (4\pi R^2) \frac{2\pi r^2}{4\pi L^2}$$

   The power that the earth emits is

$$P_e = \sigma T^4 (4\pi r^2)$$

   In equilibrium $P_i = P_3$, hence

$$\sigma T_0^4 (4\pi R^2) \frac{2\pi r^2}{4\pi L^2} = \sigma T^4 (4\pi r^2) \Rightarrow T = T_0 \sqrt{R/2L}$$

   Plugging in the numbers

$$T = 5500\sqrt{7 \cdot 10^10/2 \cdot 1.5 \cdot 10^13} = 270K$$

4. In a ferromagnetic solid at low temperatures spin waves have the dispersion relation $\omega = Ak^2$. Find the behavior of the heat capacity.

   Solution

   $\sigma(\omega)d\omega \sim \omega^{1/2}d\omega$. The heat capacity is

$$C_V = \int \frac{e^{\beta\hbar\omega}}{(e^{\beta\hbar\omega} - 1)^2} (\beta\hbar\omega)^2 \omega^{1/2} d\omega \Rightarrow C_V \sim T^{3/2}$$

5. The heat capacity of crystalline EuO varies as $C_V = aT^{3/2} + bT^3$ for $0 < T < 70K$. What state of matter is it most likely to be?

   Solution We can see the Debye result for lattice vibration, as well as the analogous term for spin waves (magnons). So it is probably a ferromagnet.

6. Assume that phonons are fermions. How would Debye's theory change in the low and high T limits?

   Solution

   In the Debye theory with bosons,

$$E = \frac{\hbar V}{\pi c^3} \int_0^{\omega_{max}} \frac{\omega^3 d\omega}{-1 + e^{\hbar\omega/kT}}$$

   and with fermions

$$E = \frac{\hbar V}{\pi c^3} \int_0^{\omega_{max}} \frac{\omega^3 d\omega}{1 + e^{\hbar\omega/kT}}$$

   . At high T bosons give a constant specific heat, while fermions zero. At low T both give $c \sim T^3$