8.08 Statistical Physics II

Random Walks and Diffusion
(Dated: May 2, 2011)

A simplified model of diffusion consists of a one-dimensional lattice, with lattice spacing a, in which an "impurity" makes a random walk from one lattice site to an adjacent one, making jumps at time intervals τ.

1. After N jumps have been made, find the probability that the atom has moved a distance d from its starting point, in the limit of large N.
2. The diffusion coefficient is defined by the differential equation $D\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t}$, where f is the concentration of the impurity. Find an expression for D in the model described above.

Solution:

1. Suppose that the particle made N_+ steps forward and N_- steps backward. The distance traveled is $d = a(N_+ - N_-) = as$ We have the relations $N_+ + N_- = n$ and $N_+ - N_- = s$. By solving these we get $N_+ = (n+s)/2$ and $N_- = (n-s)/2$.

   The probability of traveling a distance d is the probability of making N_+ jumps to the right, so

   $$P(d) = \frac{N!}{N_+!N_-!} = \frac{N!}{(N+s)/2!(N-s)/2!}$$

   Note that this is un-normalized. We will normalize it when we go to the continuum limit.

   Using Stirling's approximation we get

   $$lnP(d) = Nln2 - \frac{1}{2}(N+s)ln(1+s/N) - \frac{1}{2}(N-s)ln(1-s/N)$$

   $P(d)$ will be sharply peaked around $s = 0$, so we Taylor expand around $s/N = 0$ and get

   $$P(d) = 2^N e^{-s^2/2N}$$

   By setting $x = as$ and normalizing ww get

   $$P(x) = \frac{1}{a\sqrt{2\pi N}} w^{-x^2/2Na^2}$$

2. The number of jumps is the time over the time it takes to make one jump, $N = t/\tau$. So

   $$P(x,t) = \sqrt{\frac{\tau}{2\pi a^2 t}} e^{-x^2\tau/2a^2 t}$$

   The density of concentration of impurities in the lattice is $f(x,t)$. If we start with M impurities at time $t = 0$ in positions $x_0^1, x_0^2, ..., x_0^M$ then $f(x,t) = \sum_{i=1}^M P(x - x_0^i, t)$.

   From linearity substitute $P(x)$ into

   $$D\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial t}$$

   and get

   $$D = a^2/2\tau$$

3. Compute the ratio of thermal conductivity of helium gas at $P = 0.1atm$ and $300K$ to that at $P = 0.5atm$ and $300K$. Compute also the ratio of viscosities at the two pressures

   Solution: The rate of energy transfer through a unit area in the plane $Z = const$ is proportional to

   $$(nv)(-\lambda dE/dT) = (nv\lambda dE/dT)(-dT/dZ)$$

   The first factor is the flux of molecules in either direction across the plane. The second term is the difference in average molecular energy in a distance of one mean free path (λ). The coefficient of $(-dT/dZ)$ is the heat conductivity, $K \propto nv\lambda C$, where C is the molecular specific heat.

   The mean free path is $\lambda = 1/n\sigma$, where σ is the collision cross section. In addition, $C = 3k/2$ for a monoatomic gas and $v = \sqrt{3kT/M}$. Finally

   $$K \propto \sqrt{T}$$

   independent of pressure, so the desired ratio is 1.

   The viscosity may be calculated by considering the net transverse momentum transfer across a unit area of a plane $Z = const$, in unit time. In the same notation, this is proportional to $-(nv)(m\lambda dU/dZ)$, where m is the molecular mass and u is the transverse velocity. By definition, the coefficient of du/dZ is the viscosity η. Thus

   $$\eta \propto \sqrt{T}$$

   independent of pressure, so the ratio is 1. Furthermore

   $$\eta/K = const$$

4. In a hot plasma, all the atoms may be regarded as completely ionized. Although the ions have long-range forces due to Coulomb interactions, macroscopically the plasma is electrically neutral. This suggests that the Coulomb interactions are screened, and so become short-range. Estimate this range, making suitable approximations.

   Solution:

   The electrical potential in the vicinity of an ion is $\phi(r)$. The density near the ion is

   $$n(r) = ne^{-e\phi(r)/kT}$$

   Poisson's equation is

   $$\nabla^2\phi(r) = -4\pi\sum_\alpha e_\alpha n_\alpha$$

   For a hot plasma

   $$n_\alpha(r) = n_\alpha(1 - e_\alpha\phi/kT)$$

   Substituting into the Poisson equation

   $$\nabla^2\phi(r) = 4\pi\left(\sum_\alpha \frac{n_\alpha e_\alpha^2}{kT}\right)\phi = \kappa^2\phi$$

   which has a solution

   $$\phi(r) = e_{ion}\frac{e^{-\kappa r}}{r}$$

   κ^{-1} is a characteristic length scale of the plasma and is called the Debye radius.