8.08 Statistical Physics II

Stochastic Processes
(Dated: May 9, 2011)

1. An LC circuit is used as a thermometer by measuring the noise voltage across an inductor and capacitor in parallel. Find the relation between the rms noise voltage and the absolute temperature T.

   Solution:

$$H = \frac{L}{2}(\frac{dQ}{dt})^2 + \frac{1}{2C}q^2$$

$\omega = (LC)^{-1/2}$. $E_n = \hbar\omega(n + 1/2)$. The average energy is

$$U = \frac{\sum E_n e^{-E_n/kT}}{\sum e^{-E_n/kT}} = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{exp(\hbar\omega/hT) - 1}$$

The energy is also given by

$$\frac{U}{2} =< CV^2/2 >=< LI^2/2 >$$

$$< V^2 >= \frac{\hbar\omega}{2C} coth(\hbar\omega/2kT)$$

and

$$< I^2 >= \frac{\hbar\omega}{2L} coth(\hbar\omega/2kT)$$

In the classical limit $kT \gg \hbar\omega$ these reduce to

$$< V^2 >= kT/C \quad < I^2 >= kT/L$$

In the limit $kT \ll \hbar\omega$

$$< V^2 >= \hbar\omega/2C \quad < I^2 >= \hbar\omega/2I$$

2. Consider an oil drop of radius $0.0001cm$ in a gas of viscocity 180 micropoise and temperature 27^oC. What is the rms displacement of this drop after $10sec$? Neglect gravitational effects.

   Solution:

$$M\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} + F(t)$$

where $\beta = 6\pi R\eta$ and F is the force due to fluctuations.

$$\frac{M}{2}\frac{d^2}{dt^2}x^2 - M\dot{x}^2 = -\frac{\beta}{2}\frac{d}{dt}x^2 + xF(t)$$

Taken an ensemble average

$$\frac{1}{2}\frac{d}{dt}\left\langle \frac{d}{dt}x^2 \right\rangle - \frac{kT}{2} = -\frac{\beta}{2M}\left\langle \frac{d}{dt}x^2 \right\rangle$$

Integrating we get

$$\left\langle \frac{d}{dt} x^2 \right\rangle = \frac{2kT}{\beta} + Ce^{\frac{-\beta t}{M}}$$

$$< x^2 >= 2kTt/6\pi R\eta$$

$$< r^2 >=< x^2 > + < y^2 > + < z^2 >= kTt\pi R\eta$$

Substituting the values given we get

$$\sqrt{< r^2 >} = 1.7 \cdot 10^{-3} cm$$

3. Calculate the drag force on a disk of radius R moving with constant velocity V (perpendicular to the plane of the disk) in a rarefied gas of density n that is in thermal equilibrium at temperature T. Assume that the gas molecules collide elastically with the disk, that the speed of the disk is slow compared with the average molecular speed, and that the disk is large compared to a molecule, but small compared to the mean free path of the molecules.

   Solution:

$$\rho(v) = n(m/2\pi kT)^{3/2} exp(-mv^2/2kT)$$

The distribution in the x direction is

$$\rho(v_x) = \int\int dv_y dv_z \rho(v) = n(m/2\pi kT)^{1/2} exp(-mv_x^2/2kT)$$

The change in velocity of a particle is

$$\Delta v = -2(v_x - V)$$

The impulse imparted to the disk is $2m(v_x - V)$. The number of molecules that will collide with the area πR^2 in unit time is

$$N(v_x) = \pi R^2 \rho(v_x)(v_x - V)$$

The total impulse per unit time with particles colliding with the back of the disk is

$$I_1 = \int_V^{\infty} dv_x 2\pi m R^2 \rho(v_x)(v_x - V)^2$$

The total impulse per unit time with particles colliding with the back of the disk is

$$I_2 = -\int_{-\infty}^{V} dv_x 2\pi m R^2 \rho(v_x)(v_x - V)^2$$

The net impulse per unit time (force) is

$$F + I_1 + I_2 = 2\pi m R^2 (\int_V^\infty dv_x \rho(v_x)(v_x - V)^2 - \int_{-\infty}^V dv_x \rho(v_x)(v_x - V)^2)$$

$$F = -2\pi m R^2 \{\int_{-V}^V du \rho(u)(V+u)^2 + \int_V^\infty du \rho(u)[(V+u)^2 - (V-u)^2]\}$$

For the first integral we assume that the speed of the disk is much smaller than the average molecular speed. We approximate the Boltzman factor by unity.

$$F = -2\pi m R^2 n \sqrt{m/2\pi kT} \{8V^3/3 + (4kT/m)V exp(-mV^2/2kT)\}$$

By the assumption $mV^2 \ll kT$

$$F = -\sqrt{2\pi mkT} 4nR^2 V (1 + mV^2/6kT + ...)$$

4. At the initial instant $t = 0$ a gas occupies the half space $x < 0$. Neglecting collisions, determine the density distribution at subsequent instants.

   Solution:

$$\frac{\partial f}{\partial t} + v \frac{\partial f}{\partial r} = 0$$

In general $f = f(r - vt, v)$ and with the given initial condition $f_0 = f_0(v)$ for $v_x > x/t$ and $f = 0$ for $v_x < x//t$ where f_0 is the Maxwellian distribution.

The gas density is

$$N(t,x) = \int_\infty^\infty \int_\infty^\infty \int_x /t^\infty f_0(v) m^3 dv_x dv_y dv_z = \frac{1}{2} N_0 [1 - \Phi(\frac{x}{t} \sqrt(m/2T))$$

where

$$\Phi(\xi) = \frac{2}{\sqrt{\pi}} \int_0^\xi e^{-y^2} dy$$